11.1. Formal adjoint

Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with smooth boundary and

$$Lu := \sum_{i,j=1}^{n} a_{ij}\partial_i\partial_j u + \sum_{i=1}^{n} b_i\partial_i u + cu$$

where $a_{ij} \in C^2(\overline{\Omega})$, $b_i \in C^1(\overline{\Omega})$ and $c \in C^0(\overline{\Omega})$ for $i, j = 1, \ldots, n$. The formal adjoint operator of L is defined by

$$L^*v := \sum_{i,j=1}^{n} \partial_i\partial_j(a_{ij}v) - \sum_{i=1}^{n} \partial_i(b_i v) + cv.$$

Prove that

$$\int_\Omega v(Lu)\,\mathrm{d}x = \int_\Omega (L^*v)u\,\mathrm{d}x$$

where $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $v \in W^{2,q}(\Omega) \cap W_0^{1,p}(\Omega)$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Hint: Use the divergence theorem and Lemma 3 saying that smooth functions vanishing on the boundary are dense in both spaces.

11.2. Actual adjoint Take the same assumptions on Ω, L, L^* as in 11.1. Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and assume the ellipticity condition, i.e. that there is $\delta > 0$ with

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \delta\,|\xi|^2$$

where $\xi \in \mathbb{R}^n$, $x \in \overline{\Omega}$.

Now consider $L : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \subset L^p(\Omega) \to L^p(\Omega)$ as an unbounded linear operator with dense domain $\mathrm{dom}(L) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ on the Banach space $X = L^p(\Omega)$.

(a) Prove that L has a closed graph in $X \times X$.

Hint: Elliptic regularity!

(b) Identify the dual space X^* with $L^q(\Omega)$. Prove that the formal adjoint operator $L^* : W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \to L^q(\Omega)$ is the functional analytic dual operator[1] of L.

Hint: Elliptic regularity!

[1] As in Definition 6.2.1. of the script of FA I.

(c) Assume $p = q = 2$ and $b_i = \sum_{j=1}^n \partial_j a_{ij}$ for all $i = 1, \dots, n$. Prove that L is a self-adjoint unbounded operator on $L^2(\Omega)$.

(d) Same assumptions as in (c) plus $c = 0$. Prove that the spectrum of L in (c) is contained in $\mathbb{R}_-$.

Hint: Use 6.3.12 (v) of FA I script.

Note that this gives for example $\lambda \geq 0$, that $\Delta - \lambda : W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \to L^2(\Omega)$ is bijective, which solves the Dirichlet problem with zero on a smooth boundary for the 'Helmholtz equation' with $p = 2$.

11.3. Delta squared. Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with smooth boundary, $1 < p < \infty$. Set

$$\Gamma := \{u \in W^{4,p}(\Omega) \cap W_0^{1,p}(\Omega) : \Delta u \in W_0^{1,p}(\Omega)\}.$$

(a) Prove that $\Delta^2 = \Delta \circ \Delta : \Gamma \to L^p(\Omega)$ is bijective.

(b) Show that for every $u \in \Gamma$, $f \in L^p(\Omega)$, show that

$$\int_\Omega u \Delta^2 \varphi \, \mathrm{d}x = \int_\Omega f \varphi \, \mathrm{d}x \tag{1}$$

for every $\varphi \in C^\infty(\overline{\Omega})$ such that $\varphi|_{\partial\Omega} = 0$ and $\Delta\varphi|_{\partial\Omega} = 0$.

(c) For $u, f \in L^p(\Omega)$ satisfying (1) for every $\varphi \in C^\infty(\overline{\Omega})$ such that $\varphi|_{\partial\Omega} = 0$ and $\Delta\varphi|_{\partial\Omega} = 0$, prove that $u \in \Gamma$.

Hint: Try to squeeze u into the situation of higher regularity (Theorem 5).

Note: It can be observed while solving the exercise that Δ can be replaced by L in divergence form with coefficients $a_{ij} \in C^{2+1}(\overline{\Omega})$.

11.4. Maximum principle: easy case Let $\Omega \subset \mathbb{R}^n$ be a bounded, open domain and let $u \in C^2_{loc}(\Omega) \cap C(\overline{\Omega})$. Let L be an elliptic operator as in 11.2, with[2] $a_{ij}, b_i, c \in C^0(\overline{\Omega})$ for $i, j = 1, \dots, n$.

(a) Assume that $c < 0$ and $Lu \geq 0$ in Ω. Then prove that

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+$$

where $u^+(x) := \max(0, u(x))$ is the positive part of u.

[2] The assumptions on the coefficients are actually too strong.

(b) Find u with $u < 0$, $\Delta u \geq u$ and $\max_{\overline{\Omega}} u > \max_{\partial\Omega} u$. Deduce that u^+ cannot be replaced by u in (a).

Please hand in your solutions for this sheet by Tuesday 17/05/2016 to the pigeonhole of your teaching assistant in F28.