11.1. Formal adjoint Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with smooth boundary and

$$Lu := \sum_{i,j=1}^{n} a_{ij}\partial_i\partial_j u + \sum_{i=1}^{n} b_i\partial_i u + cu$$

where $a_{ij} \in C^2(\overline{\Omega})$, $b_i \in C^1(\overline{\Omega})$ and $c \in C^0(\overline{\Omega})$ for $i, j = 1, \dots, n$ and $u \in W^{2,p}(\Omega)$. The formal adjoint operator of L is defined by

$$L^*v := \sum_{i,j=1}^{n} \partial_i\partial_j(a_{ij}v) - \sum_{i=1}^{n} \partial_i(b_i v) + cv.$$

where $v \in W^{2,q}(\Omega)$. Prove that

$$\int_\Omega v(Lu)\ \mathrm{d}x = \int_\Omega (L^*v)u\ \mathrm{d}x$$

where $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $v \in W^{2,q}(\Omega) \cap W_0^{1,p}(\Omega)$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Hint: Use the divergence theorem and Lemma 3 saying that smooth functions vanishing on the boundary are dense in both spaces.

Solution: We have for $u, v \in C^\infty(\overline{\Omega})$ with $u|_{\partial\Omega} = v|_{\partial\Omega} = 0$. Let $n : \partial\Omega \to \mathbb{R}^n$ be the outward painting normal. Then we get

$$\begin{aligned}
\int_\Omega v(a_{ij}\partial_i\partial_j u)\ \mathrm{d}x &= -\int_\Omega \partial_i(a_{ij}v)\partial_j u\ \mathrm{d}x + \int_{\partial\Omega} v a_{ij}\partial_j v n_i\ \mathrm{d}\sigma \\
&= \int_\Omega \partial_j\partial_i(a_{ij}v)u\ \mathrm{d}x + \int_{\partial\Omega} \partial_j(a_{ij}v)u\ \mathrm{d}\sigma \\
&= \int_\Omega (\partial_j\partial_i(a_{ij}v))u\ \mathrm{d}x \\
\int_\Omega v(b_i\partial_i u)\ \mathrm{d}x &= -\int_\Omega \partial_i(b_i v)u\ \mathrm{d}x + \int_{\partial\Omega} b_i vu\ \mathrm{d}\sigma \\
&= -\int_\Omega (\partial_i(b_i v))u\ \mathrm{d}x \\
\int_\Omega v(cu)\ \mathrm{d}x &= \int_\Omega (cv)u\ \mathrm{d}x
\end{aligned}$$

for all $i, j = 1, \dots, n$ and where we use the integration by part formula repeatedly. The boundary terms vanished because either v or u appeared in the expression for the boundary integral. Suming over all the $i, j = 1, \dots, n$, we get the wanted identity for $u, v \in C^\infty(\overline{\Omega})$ with $u|_{\partial\Omega} = v|_{\partial\Omega} = 0$. As this represents a dense subset of both function spaces and as we can apply Hölder, the identity still holds for $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and $v \in W^{2,q}(\Omega) \cap W_0^{1,p}(\Omega)$.

11.2. Actual adjoint Take the same assumptions on Ω, L, L^* as in 11.1. Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and assume the ellipticity condition, i.e. that there is $\delta > 0$ with

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \delta\,|\xi|^2$$

where $\xi \in \mathbb{R}^n$, $x \in \overline{\Omega}$.

Now consider $L : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \subset L^p(\Omega) \to L^p(\Omega)$ as an unbounded linear operator with dense domain $\mathrm{dom}(L) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ on the Banach space $X = L^p(\Omega)$.

(a) Prove that L has a closed graph in $X \times X$.

Hint: Elliptic regularity!

(b) Identify the dual space X^* with $L^q(\Omega)$. Prove that the formal adjoint operator $L^* : W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \to L^q(\Omega)$ is the functional analytic dual operator[1] of L.

Hint: Elliptic regularity!

(c) Assume $p = q = 2$ and $b_i = \sum_{j=1}^{n} \partial_j a_{ij}$ for all $i = 1, \ldots, n$. Prove that L is a self-adjoint unbounded operator on $L^2(\Omega)$.

(d) Same assumptions as in (c) plus $c = 0$. Prove that the spectrum of L in (c) is contained in $\mathbb{R}_-$.

Hint: Use 6.3.12 (v) of FA I script.

Note that this gives for example $\lambda \geq 0$, that $\Delta - \lambda : W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \to L^2(\Omega)$ is bijective, which solves the Dirichlet problem with zero on a smooth boundary for the 'Helmholtz equation' with $p = 2$.

Solution:

(a) Let us take a sequence $u_l \in X$ such that $(u_l, L(u_l))$ converges in $X \times X$ to $(u, v) \in X \times X$. We need to prove that $v = Lu$. Now for any $\varphi \in C^\infty(\Omega)$ with $\varphi|_{\partial\Omega} = 0$, we get by 11.1,

$$\int_\Omega uL^*\varphi = \lim_{l\to\infty} \int_\Omega u_l L^*\varphi = \lim_{l\to\infty} \int_\Omega Lu_l\varphi = \int_\Omega v\varphi.$$

Thus we land in the assumptions of[2] Theorem 6 (ii) with $k = 0$, and so we find that $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and that $Lu = v$. Alternatively, you use (b) to have $(L^*)^* = L$ and the fact that the dual of every operator is closed as in 6.2.2 (i) of the FAI script.

[1] As in Definition 6.2.1. of the script of FA I.
[2] If you missed the lecture, there are my handwritten notes on the website for this chapter.

(b) We will locally introduce the notation $L^\vee$ for the functional analytic dual of L. First we need to determine the domain $\operatorname{dom}(L^\vee)$ of $L^\vee$. By definition, we have

$$\operatorname{dom}(L^\vee) = \left\{ v \in L^q(\Omega) : \begin{array}{c} \text{there exists a constant } c \geq 0 \text{ such that} \\ |\langle v, Lu\rangle| \leq c \, \|u\|_X \text{ for all } u \in \operatorname{dom}(L). \end{array} \right\}.$$

Thus take $u \in \operatorname{dom}(L)$ and $v \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$, and observe by 11.1, that

$$|\langle v, Lu\rangle| = \left| \int_\Omega vLu \right| = \left| \int_\Omega (L^* v)u \right| \leq \|L^* v\|_{L^q(\Omega)} \|u\|_X$$

by Hölder. Hence, $W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \subset \operatorname{dom}(L^\vee)$. On the other hand, given any $v \in \operatorname{dom}(L^\vee)$, we get an element $L^\vee v \in L^q(\Omega)$ such that

$$\int_\Omega (L^\vee v)u = \langle L^\vee v, u\rangle = \langle v, Lu\rangle = \int_\Omega vLu \tag{1}$$

for all $u \in \{u \in C^\infty(\overline{\Omega}) : u|_{\partial\Omega} = 0\} \subset \operatorname{dom}(L)$. Now a short calculation shows that the coefficients of L^* are

$$a_{ij}^* = a_{ij}, \qquad b_i^* = -b_i + 2\sum_{j=1}^n \partial_j a_{ij}, \qquad c^* = c - \sum_{i=1}^n \partial_i b_i + \sum_{i,j=1}^n \partial_i \partial_j a_{ij} \tag{2}$$

for $i, j = 1, \ldots, n$ and the coefficients have the same regularity as the coefficients of L. Iterating this formula, we have that the coefficients of $(L^*)^*$, we find

$$a_{ij}^{**} = a_{ij}, \qquad b_i^{**} = b_i, \qquad c^{**} = c.$$

So $(L^*)^* = L$ and therefore (1) is simply the statement that v is a weak solution to $L^* v = L^\vee v$ with zero boundary. Therefore, by theorem 6 (ii) again, we get $v \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ and $L^* v = L^\vee v$. So we also have $\operatorname{dom}(L^\vee) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ and $L^* = L^\vee$.

(c) From the previous point, we have that under these conditions on the coefficients, by (2), $L^* = L$. As $X^* = X$ under the same pairing, we have that $L^\vee = L$.

(d) As L is self-adjoint, we have $\sigma(L) \subset \mathbb{R}$. Note also that by our assumptions, $L = \sum_{i,j=1}^n \partial_j (a_{ij} \partial_i u)$ is in (minus) the divergence form, so we get

$$\begin{aligned}
\sup \sigma(L) &= \sup\{\langle u, Lu\rangle : u \in \operatorname{dom}(L), \|u\|_X = 1\} \\
&= \sup\{\int_\Omega uLu : u \in \operatorname{dom}(L), \|u\|_X = 1\} \\
&= \sup\{- \sum_{i,j=1}^n \int_\Omega (\partial_j u) a_{ij} \partial_i u : u \in \operatorname{dom}(L), \|u\|_X = 1\} \\
&\leq \sup\{-\delta \int_\Omega |\nabla u|^2 : u \in \operatorname{dom}(L), \|u\|_X = 1\} \leq 0.
\end{aligned}$$

where the penultimate inequality uses ellipticity.

11.3. Delta squared. Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with smooth boundary, $1 < p < \infty$. Set

$$\Gamma := \{u \in W^{4,p}(\Omega) \cap W_0^{1,p}(\Omega) : \Delta u \in W_0^{1,p}(\Omega)\}.$$

(a) Prove that $\Delta^2 = \Delta \circ \Delta : \Gamma \to L^p(\Omega)$ is bijective.

(b) Show that for every $u \in \Gamma$, $f \in L^p(\Omega)$, show that

$$\int_\Omega u\Delta^2\varphi \, \mathrm{d}x = \int_\Omega f\varphi \, \mathrm{d}x \tag{3}$$

for every $\varphi \in C^\infty(\overline{\Omega})$ such that $\varphi|_{\partial\Omega} = 0$ and $\Delta\varphi|_{\partial\Omega} = 0$.

(c) For $u, f \in L^p(\Omega)$ satisfying (3) for every $\varphi \in C^\infty(\overline{\Omega})$ such that $\varphi|_{\partial\Omega} = 0$ and $\Delta\varphi|_{\partial\Omega} = 0$, prove that $u \in \Gamma$.

Hint: Try to squeeze u into the situation of higher regularity (Theorem 5).

Note: It can be observed while solving the exercise that Δ can be replaced by L in divergence form with coefficients $a_{ij} \in C^{2+1}(\overline{\Omega})$.

Solution:

(a) We use the fact that $\Delta : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to L^2(\Omega)$ is injective to conclude for $u \in \Gamma$, that

$$(\Delta^2(u) = 0 \text{ and } u \in \Gamma) \Rightarrow (\Delta u = 0 \text{ and } u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)) \Rightarrow u = 0.$$

We use the fact that $\Delta : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to L^2(\Omega)$ is surjective to conclude that for $f \in L^p(\Omega)$ there is $g \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that $\Delta g = f$. By higher regularity (Theorem 5), we also get there is $u \in W^{4,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that $\Delta u = g \in W_0^{1,p}(\Omega)$. Hence, $u \in \Gamma$ and $\Delta^2 u = f$.

(b) As for $L^* = \Delta$, we get by 11.1 that

$$\int_\Omega f\varphi = \int_\Omega \Delta(\Delta u)\varphi = \int_\Omega \Delta u \Delta\varphi = \int_\Omega u\Delta^2\varphi$$

where in the last equality we used the fact that $\Delta\varphi|_{\partial\Omega} = 0$.

(c) For $\psi \in C^\infty(\overline{\Omega})$ with $\psi|_{\partial\Omega} = 0$, we can choose $v, \varphi \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ with $\Delta\varphi = \psi$ and $\Delta v = f$. By higher regularity (Theorem 5), we know that $\varphi \in C^\infty(\overline{\Omega})$ and we have $\varphi|_{\partial\Omega} = 0$ and $\Delta\varphi|_{\partial\Omega} = 0$. Therefore, we get

$$\int_\Omega u\Delta\psi = \int_\Omega u\Delta^2\varphi = \int_\Omega f\varphi = \int_\Omega \Delta v\varphi = \int_\Omega v\Delta\varphi = \int_\Omega v\psi$$

where the first, third and last equality follow by definition, and the second on follows by (3). The penultimate equality follows by 11.1. Thus, by higher regularity (Theorem 5), we get that $u \in W^{4,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and $\Delta u = v \in W_0^{1,p}(\Omega)$. So we actually have $u \in \Gamma$.

11.4. Maximum principle: easy case Let $\Omega \subset \mathbb{R}^n$ be a bounded, open domain and let $u \in C^2_{loc}(\Omega) \cap C(\overline{\Omega})$. Let L be an elliptic operator as in 11.2, with[3] $a_{ij}, b_i, c \in C^0(\overline{\Omega})$ for $i, j = 1, \dots, n$.

(a) Assume that $c < 0$ and $Lu \geq 0$ in Ω. Then prove that

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+$$

where $u^+(x) := \max(0, u(x))$ is the positive part of u.

(b) Find u with $u < 0$, $\Delta u \geq u$ and $\max_{\overline{\Omega}} u > \max_{\partial\Omega} u$. Deduce that u^+ cannot be replaced by u in (a).

Solution:

(a) As $\overline{\Omega}$ is compact, u attains a maximum. Assume by contradiction that there is $x_0 \Omega$ where $u(x_0) = \max_{\overline{\Omega}} u > 0$. Then at this point the Hessian matrix $H := [\partial_i \partial_j u(x_0)]_{i,j=1,\dots,n}$ is semi-negative definite. As $A = [a_{ij}(x_0)]_{i,j=1,\dots,n}$ is positive definite, AH is semi-negative definite and symmetric. Therefore, $\sum_{i,j=1}^n a_{ij}(x_0)\partial_i\partial_j u = \mathrm{tr}(AH) \leq 0$. Also $\nabla u(x_0) = 0$ as x_0 is a critical point. Hence, we have

$$Lu(x_0) = \mathrm{tr}(AH) + 0 + c(x_0)u(x_0) \leq c(x_0)u(x_0) < 0$$

contradicting the fact that $Lu \geq 0$. This proves the wanted inequality, as the expression is void in case $\max_{\overline{\Omega}} u < 0$.

(b) Take $u(x) = -2n - |x|^2$ and $\Omega = B_1(0)$. Then $u \leq -2n < 0$, $\Delta u(x) = -2n \geq -2n - |x|^2 = u(x)$ for all $x \in B_1(0)$ and $-2n = \max_{\overline{\Omega}} u > -2n - 1 = \max_{\partial\Omega} u$. However, $\max_{\overline{\Omega}} u = -2n \leq \max_{\partial\Omega} u^+ = 0$ still holds.

[3] The assumptions on the coefficients are actually too strong.