12.1. Regularity on $\mathbb{R}^n$. Let $1 < p < \infty$. Prove that for $f \in L^p(\mathbb{R}^n)$, $u \in L^p(\mathbb{R}^n)$ with

$$\int_{\mathbb{R}^n} u\Delta\varphi = \int_{\mathbb{R}^n} f\varphi$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$. Then $u \in W^{2,p}(\mathbb{R}^n)$ and $\Delta u = f$.

Hint: Use Theorem 7 on local regularity with a cubes of radius $\frac{1}{2}$ in a box of radius 1 and translate.

12.2. The heat kernel. Let $K_t(x) := \frac{1}{(4\pi t)^{n/2}} e^{\frac{-|x|^2}{4t}}$ for $x \in \mathbb{R}^n$ and $t > 0$.

(a) Prove that $\partial_t K_t = \Delta K_t$, i.e. the kernel of the heat equation is a solution of the heat equation.

(b) Prove that $I := \int_{\mathbb{R}^n} K_t \, \mathrm{d}x = 1$ for $t > 0$. **Hint:** Calculate I^2 instead, use spherical coordinates and use $\omega_{2n} = \frac{2\pi^n}{(n-1)!}$.

(c) Prove that $K_{t+s} = K_t * K_s$ for all $s, t > 0$. **Hint:** Go to the Fourier side.

(d) Prove that $\lim_{t\to 0} \|K_t * u - u\|_{L^p(\mathbb{R}^n)} = 0$ for all $1 \le p \le \infty$ and for all $u \in C_0(\mathbb{R}^n)$. **Hint:** Start with $p = \infty$.

(e) Conclude that for $1 \le p < \infty$, $\lim_{t\to 0} \|K_t * u - u\|_{L^p(\mathbb{R}^n)} = 0$ for all $u \in L^p(\mathbb{R}^n)$. **Hint:** Use Banach-Steinhaus. (2.1.5)[1]

(f) Conclude that $S(t) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for $t \ge 0$ given by

$$S(t)u_0 := \begin{cases} K_t * u_0 & \text{for} \quad t > 0 \\ u_0 & \text{for} \quad t = 0 \end{cases}$$

is a strongly continuous semigroup. (7.1.1) Prove that this is a contracting (7.2.9) and self-adjoint (7.3.10, $p = 2$) strongly continuous semigroup.

(g) For $u_0 \in L^p(\mathbb{R}^n)$ prove that $u : (0, \infty) \times \mathbb{R}^n \to \mathbb{R} : (t, x) \mapsto (S(t)u_0)(x)$ is a smooth solution of the heat equation $\partial_t u = \Delta u$ with $\lim_{t\to 0} \|u(t, \cdot) - u_0\|_{L^p(\mathbb{R}^n)} = 0$.

(h) Determine the infinitesimal generator (7.1.9) of S for $1 < p < \infty$. **Hint:** Use Lemma 5.1.9 and Exercise 12.1 to determine the domain.

(i) Prove that the heat equation with initial values in $W^{2,p}(\mathbb{R}^n)$ is a well-posed Cauchy problem. **Hint:** Use 7.2.2.

[1] All such references in this exercise sheet are to the FA I script.

12.3. The heat equation on a bounded domains. Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain with smooth boundary. Take L to be a divergence form elliptic operator i.e. $Lu = \sum_{i,j=1}^n \partial_i(a_{ij}\partial_j u)$ with $a_{ij} = a_{ji} \in C^\infty(\overline{\Omega})$ and there is $\delta > 0$ with $\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \delta\,|\xi|^2$ for all $\xi \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$.

(a) Prove that $L : W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ is the infinitesimal generator of a contraction strongly continuous semigroup S which is also self-adjoint.

Hint: Use Lumer–Phillips Theorem (7.2.11) and the theorem on self-adjoint semigroups (7.3.10) together with exercise 11.2. (c).

(b) S cannot be extended into a strongly continuous group.

Hint: Use the theorem on strongly continuous groups (7.2.4) and the corollary to Hille–Yoshida (7.2.8) together with the fact that L is bijective and $L^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is compact together with the spectral theory for compact operators (5.2.7) and self-adjoint operators (5.3.16).

(c) Prove that for $u_0 \in L^2(\mathbb{R}^n)$, $S(t)u_0$ is smooth and vanishes on the boundary for every $t > 0$.

Hint: Use the theorem on strongly continuous analytic semigroups 7.4.2 and example 7.4.5 on analyticity of self-adjoint strongly continuous semigroups.

Please hand in your solutions for this sheet by Monday 23/05/2016.