12.1. Regularity on $\mathbb{R}^n$. Let $1 < p < \infty$. Prove that for $f \in L^p(\mathbb{R}^n)$, $u \in L^p(\mathbb{R}^n)$ with

$$\int_{\mathbb{R}^n} u\Delta\varphi = \int_{\mathbb{R}^n} f\varphi$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$. Then $u \in W^{2,p}(\mathbb{R}^n)$ and $\Delta u = f$.

Hint: Use Theorem 7 on local regularity with a cubes of radius $\frac{1}{2}$ in a box of radius 1 and translate.

Solution: We directly obtain from (i) that $u \in W^{k+2,p}_{loc}(\mathbb{R}^n)$ and $\Delta u = f$. Now define $\|x\|_\infty := \max_{i=1,\dots,n} |x_i|$ and take $K = \overline{B}^{\|\cdot\|_\infty}_{1/2}(0)$ and $\Omega = B^{\|\cdot\|_\infty}_1(0)$ in (ii) to get the estimate for $C > 0$

$$\|u\|_{W^{2,p}(K)} \leq C(\|\Delta u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}). \tag{1}$$

Now define τ_α for $\alpha \in \mathbb{Z}^n$ to be the translation by this vector. As all the norms in (1) are translation invariant, we have that

$$\|u\|_{W^{2,p}(\tau_\alpha K)} \leq C(\|f\|_{L^p(\tau_\alpha \Omega)} + \|u\|_{L^p(\tau_\alpha \Omega)})$$

for all $\alpha \in \mathbb{Z}^n$. (Same constant C>0 !) Now the union of $\tau_\alpha K$ for $\alpha \in \mathbb{Z}^n$ covers every point outside of a set of measure zero exactly once and the union of $\tau_\alpha K$ for $\alpha \in \mathbb{Z}^n$ covers every point outside of a set of measure zero exactly 2^n times. Here the set of measure zero is the countable union of faces of the cubes $\tau_\alpha K$. Thus we get the estimate

$$\begin{aligned}\|u\|_{W^{2,p}(\mathbb{R}^n)} &= \sum_{\alpha\in\mathbb{Z}^n} \|u\|_{W^{2,p}(\tau_\alpha K)} \\ &\leq C \sum_{\alpha\in\mathbb{Z}^n} (\|f\|_{L^p(\tau_\alpha\Omega)} + \|u\|_{L^p(\tau_\alpha\Omega)}) = 2^n C(\|f\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)}) < \infty\end{aligned}$$

So $u \in W^{2,p}(\mathbb{R}^n)$.

Note: This last estimate could have also been obtained by a combination of Calderòn–Zygmund and Gagliardo-Nirenberg together with an approximation by smooth functions with compact support.

12.2. The heat kernel. Let $K_t(x) := \frac{1}{(4\pi t)^{n/2}} e^{\frac{-|x|^2}{4t}}$ for $x \in \mathbb{R}^n$ and $t > 0$.

(a) Prove that $\partial_t K_t = \Delta K_t$, i.e. the kernel of the heat equation is a solution of the heat equation.

(b) Prove that $I := \int_{\mathbb{R}^n} K_t \, \mathrm{d}x = 1$ for $t > 0$. **Hint:** Calculate I^2 instead, use spherical coordinates and use $\omega_{2n} = \frac{2\pi^n}{(n-1)!}$.

(c) Prove that $K_{t+s} = K_t * K_s$ for all $s, t > 0$. **Hint:** Go to the Fourier side.

(d) Prove that $\lim_{t\to 0} \|K_t * u - u\|_{L^p(\mathbb{R}^n)} = 0$ for all $1 \le p \le \infty$ and for all $u \in C_0(\mathbb{R}^n)$. **Hint:** Start with $p = \infty$.

(e) Conclude that for $1 \le p < \infty$, $\lim_{t\to 0} \|K_t * u - u\|_{L^p(\mathbb{R}^n)} = 0$ for all $u \in L^p(\mathbb{R}^n)$. **Hint:** Use Banach-Steinhaus. (2.1.5)[1]

(f) Conclude that $S(t) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for $t \ge 0$ given by

$$S(t)u_0 := \begin{cases} K_t * u_0 & \text{for} \quad t > 0 \\ u_0 & \text{for} \quad t = 0 \end{cases}$$

is a strongly continuous semigroup. (7.1.1) Prove that this is a contracting (7.2.9) and self-adjoint (7.3.10, $p = 2$) strongly continuous semigroup.

(g) For $u_0 \in L^p(\mathbb{R}^n)$ prove that $u : (0, \infty) \times \mathbb{R}^n \to \mathbb{R} : (t, x) \mapsto (S(t)u_0)(x)$ is a smooth solution of the heat equation $\partial_t u = \Delta u$ with $\lim_{t\to 0} \|u(t, \cdot) - u_0\|_{L^p(\mathbb{R}^n)} = 0$.

(h) Determine the infinitesimal generator (7.1.9) of S for $1 < p < \infty$. **Hint:** Use Lemma 5.1.9 and Exercise 12.1 to determine the domain.

(i) Prove that the heat equation with initial values in $W^{2,p}(\mathbb{R}^n)$ is a well-posed Cauchy problem. **Hint:** Use 7.2.2.

Solution:

(a) We simply compute for $i = 1, \dots, n$ and $t > 0$,

$$\begin{aligned}
\partial_t K_t &= K_t\left(-\frac{n}{2t} + \frac{|x|^2}{4t^2}\right), \\
\partial_i K_t &= -K_t \frac{2x_i}{4t}, \\
\partial_i^2 K_t &= K_t\left(-\frac{2}{4t} - \frac{x_i^2}{4t^2}\right).
\end{aligned}$$

Hence, summing over i, we get $\partial_t K_t = \Delta K_t$ for all $t > 0$.

(b) Fix $t > 0$. Call $I = \sum_{\mathbb{R}^n} K_t(x) \mathrm{d}x$. We will calculate I^2 as follows

$$\begin{aligned}
(4\pi t)^n I^2 &= \int_{(\mathbb{R}^n)^2} e^{-\frac{|x|^2}{4t}} e^{-\frac{|y|^2}{4t}} \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_{\mathbb{R}^{2n}} e^{-\frac{|z|^2}{4t}} \mathrm{d}z \\
&= \omega_{2n} \int_0^\infty e^{-\frac{r^2}{4t}} r^{2n-1} \, \mathrm{d}r
\end{aligned}$$

[1] All such references in this exercise sheet are to the FA I script.

Define $I_n := \int_0^\infty e^{-\frac{r^2}{4t}} r^{2n-1} \, \mathrm{d}r$. We can inductively calculate this integral from $I_1 = 2t$. Indeed, by integration by part, we get for $n > 1$

$$\begin{aligned} I_n &= 2t(2n-2)\int_0^\infty e^{-\frac{r^2}{4t}} r^{2n-3} \, \mathrm{d}r = 2t(2n-2)I_{n-1} = \dots \\ &= (2t)^{n-1} \prod_{i=1}^{n-1} (2n-2i) I_1 = (2t)^n 2^{n-1}(n-1)! \end{aligned}$$

Thus

$$I^2 = \frac{1}{(4t\pi)^n}(2t)^n 2^{n-1}(n-1)! \, \omega_{2n} = 1.$$

(c) We calculate the Fourier transform of $K_t \in \mathcal{S}(\mathbb{R}^n)$. For $t > 0$, we get

$$\begin{aligned} \mathcal{F}(K_t)(\xi) &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} e^{-\frac{|x|^2}{4t}} \, \mathrm{d}x = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-t|\xi|^2} e^{-\left|t^{1/2} i\xi + \frac{x}{2\sqrt{t}}\right|^2} \, \mathrm{d}x \\ &= e^{-t|\xi|^2} \int_{\mathbb{R}^n} K_t(x) \, \mathrm{d}x = e^{-t|\xi|^2} \end{aligned}$$

where the third equality comes from invariance under translation of the Lebesgue measure and the last by the calculations in (b). Thus we get for $s, t > 0$ that

$$\mathcal{F}(K_{s+t}) = \mathcal{F}(K_s)\mathcal{F}(K_t).$$

Therefore, we get under Fourier inverse

$$K_{s+t} = K_s * K_t$$

for all $s, t > 0$.

(d) Take $u \in C_0(\mathbb{R}^n)$. Fix $\epsilon > 0$. Then by absolute continuity, there is $\delta > 0$ such that $|u(x-y) - u(x)| \leq \frac{\epsilon}{2}$ for $x \in \mathbb{R}^n$ and $y \in B_\delta(0)$. Thus we get for $x \in \mathbb{R}^n$, that

$$\begin{aligned} &K_t * u(x) - u(x) = \int_{\mathbb{R}^n} K_t(y)(u(x-y) - u(x)) \, \mathrm{d}y \\ &= \int_{B_\delta(0)} K_t(y)(u(x-y) - u(x)) \, \mathrm{d}y + \int_{\mathbb{R}^n \setminus B_\delta(0)} K_t(y)(u(x-y) - u(x)) \, \mathrm{d}y \\ &= I_1 + I_2 \end{aligned}$$

Now, by choice of $\delta > 0$ and $\int_{\mathbb{R}^n} K_t \, \mathrm{d}x = 1$, we have $|I_1| \leq \frac{\epsilon}{2}$. Now for I_2, we notice that

$$\int_{\mathbb{R}^n \setminus B_\delta(0)} K_t(y) \, \mathrm{d}y = \int_{\mathbb{R}^n \setminus B_r(0)} K_1(y) \, \mathrm{d}y$$

where $r = \frac{\delta}{\sqrt{t}}$. As $\int_{\mathbb{R}^n} K_1 \, \mathrm{d}x = 1$, we have that there is $R > 0$ such that if $r > R$, then

$$2 \|u\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus B_r(0)} K_1(y) \, \mathrm{d}y < \frac{\epsilon}{2}$$

and so by definition, for $0 < t < \frac{\delta^2}{R^2}$, we get $|I_2| < \frac{\epsilon}{2}$. This proves that

$$\lim_{t \to 0} \|K_t * u - u\|_{L^\infty(\mathbb{R}^n)} = 0.$$

Now let $R > 0$ such that $\operatorname{supp} u \subset B_R(0)$, then we have

$$\begin{aligned}
&\lim_{t \to 0} \|K_t * u - u\|_{L^p(\mathbb{R}^n)} \\
&= \lim_{t \to 0} \|K_t * u - u\|_{L^p(B_R(0))} + \lim_{t \to 0} \|K_t * u - u\|_{L^p(\mathbb{R}^n \setminus B_R(0))} \\
&\leq \mu(\operatorname{supp} f)^{\frac{1}{p}} \lim_{t \to 0} \|K_t * u - u\|_{L^\infty(\mathbb{R}^n)} + \lim_{t \to 0} \|K_t\|_{L^1(\mathbb{R}^n \setminus B_R(0))} \|u\|_{L^p(\mathbb{R}^n)} = 0
\end{aligned}$$

where we used Hölder's inequality and Young's inequality.

(e) As $C_c(\mathbb{R}^n)$ is dense in every $L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$ and as for $t > 0$

$$\|S(t)u\|_{L^p} \leq \|K_t\|_{L^1} \|u\|_{L^p}$$

by Young's inequality, we have that

$$\|S(t)\| \leq 1, \tag{2}$$

we can apply Banach-Steinhaus to conclude that $S(t)$ converges strongly to the identity on $L^p(\mathbb{R}^n)$.

(f) By the previous paragraph, S is strongly continuous. That it is a semi-group follows immediately from (c). That it is a contraction group follows by definition from (2). That for $p = 2$, $S(t)$ is self-adjoint, follows from $K_t(x) = K_t(-x)$ for all $t > 0$ and $x \in \mathbb{R}^n$, because then

$$\langle S(t)u, v \rangle_{L^2} = \int_{(\mathbb{R}^n)^2} u(x) K_t(x - y) v(y) = \langle u, S(t)v \rangle_{L^2}.$$

(g) That u is smooth follows from K_t being smooth. Also from (a), we get that u is solution of the heat equation and the last statement comes from the strong continuity of S.

(h) An educated guess would be that $A : \operatorname{dom}(A) \to L^p(\mathbb{R}^n)$, the infinitesimal generator of S, is equal to $\Delta : W^{2,p}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$.

We start by showing that $W^{2,p}(\mathbb{R}^n) \subset \text{dom}(A)$. So fix $u \in W^{2,p}(\mathbb{R}^n)$, $T > 0$ and by the same argument as in Analysis I, the function $x : [0,T] \to L^p(\mathbb{R}^n) : t \to \int_0^t (K_s * \Delta u)\, \mathrm{d}s$ is continuously differentiable and its derivative $\dot{x}$ is equal $\dot{x}(t) = K_t * \Delta u$. In particular, $\dot{x}(0) = \Delta u$ by strong continuity. Let $v \in L^q(\mathbb{R}^n) = (L^p(\mathbb{R}^n))^*$ with $\frac{1}{p} + \frac{1}{q} = 1$, and so

$$\left\langle v, \int_0^t K_s * \Delta u \, \mathrm{d}s \right\rangle := \int_0^t \langle v, K_s * \Delta u \rangle \, \mathrm{d}s = \lim_{\delta \to 0^+} \int_\delta^t \langle v, K_s * \Delta u \rangle \, \mathrm{d}s$$

where the last inequality follows by dominated convergence. But now we have for $\delta \leq s \leq t$ and $x \in \mathbb{R}^n$, that

$$K_s * \Delta u(x) = \Delta K_s * u(x) = (\partial_s K_s) * u(x) = \partial_s (K_s * u(x))$$

where we used (a) and usual differentiation rules for convolution products. Here we commit a minor notational abuse in noting the derivative $\dot{\gamma}(s)$ of the path $\gamma : [\delta, t] \to \mathbb{R} : s \to \langle v, K_s * u \rangle$ by $\partial_s \langle v, K_s * u \rangle$. Thus, we pursue

$$\begin{aligned} \left\langle v, \int_0^t K_s * \Delta u \, \mathrm{d}s \right\rangle &= \lim_{\delta \to 0^+} \int_\delta^t \partial_s \langle v, K_s * u \rangle \, \mathrm{d}s \\ &= \lim_{\delta \to 0^+} \langle v, K_t * u - K_\delta * u \rangle = \langle v, K_t * u - u \rangle \end{aligned}$$

where we use 5.1.9. in the second equality and Hölder inequality together with strong continuity in the last equality. As $v \in L^q(\mathbb{R}^n)$ was arbitrary, we have established by Hahn-Banach, that

$$\int_0^t K_s * \Delta u \, \mathrm{d}s = K_t * u - u$$

which immediately tells us by combining all the above that $u \in \text{dom}(A)$ and $Au = \Delta u$.

Now we still have to prove that $\text{dom}(A) \subset W^{2,p}(\mathbb{R}^n)$. So let $u \in \text{dom}(A) \subset L^p(\mathbb{R}^n)$. Then fix $\varphi \in C_0^\infty(\mathbb{R}^n)$, so in particular $\varphi \in W^{2,q}(\mathbb{R}^n)$. So by what we just established, we have $\lim_{h \to 0^+} \frac{K_h * \varphi - \varphi}{h} = \Delta \varphi$ where the limit is take in $L^q(\mathbb{R}^n)$. Therefore, by Hölder and dominated convergence, we have

$$\langle \varphi, Au \rangle := \lim_{h \to 0^+} \left\langle \varphi, \frac{K_h * u - u}{h} \right\rangle = \lim_{h \to 0^+} \left\langle \frac{K_h * \varphi - \varphi}{h}, u \right\rangle = \langle \Delta \varphi, u \rangle$$

where all the pairings are the usual $L^p - L^q$ one and the second equality also used the symmetry $K_h(x) = K_h(-x)$. Therefore, we get that u is a weak solution with $f = Au \in L^p(\mathbb{R}^n)$. Therefore, by 12.1, we have that $u \in W^{2,p}(\mathbb{R}^n)$. This ends the proof of $A = \Delta$.

(i) We know that $W^{2,p}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ and $\Delta : W^{2,p}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is closed by the usual argument of elliptic regularity used in 11.2. (a). So we can apply Phillips theorem, and conclude that it is a well-posed problem.

12.3. The heat equation on a bounded domains. Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain with smooth boundary. Take L to be a divergence form elliptic operator i.e. $Lu = \sum_{i,j=1}^n \partial_i(a_{ij}\partial_j u)$ with $a_{ij} = a_{ji} \in C^\infty(\overline{\Omega})$ and there is $\delta > 0$ with $\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \delta\,|\xi|^2$ for all $\xi \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$.

(a) Prove that $L : W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ is the infinitesimal generator of a contraction strongly continuous semigroup S which is also self-adjoint.

Hint: Use Lumer–Phillips Theorem (7.2.11) and the theorem on self-adjoint semigroups (7.3.10) together with exercise 11.2. (c).

(b) S cannot be extended into a strongly continuous group.

Hint: Use the theorem on strongly continuous groups (7.2.4) and the corollary to Hille–Yoshida (7.2.8) together with the fact that L is bijective and $L^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is compact together with the spectral theory for compact operators (5.2.7) and self-adjoint operators (5.3.16).

(c) Prove that for $u_0 \in L^2(\mathbb{R}^n)$, $S(t)u_0$ is smooth and vanishes on the boundary for every $t > 0$.

Hint: Use the theorem on strongly continuous analytic semigroups 7.4.2 and example 7.4.5 on analyticity of self-adjoint strongly continuous semigroups.

Solution:

(a) For $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, we have that

$$\langle u, Lu\rangle = -\int_\Omega \partial_i u \; a_{ij} \; \partial_j u = -B(u,u) \leq 0.$$

and due to Exercise 11.2. (d), we know that for all $\lambda > 0$, we have $L - \lambda \mathbb{1}$ is bijective, so in particular has a dense image. Therefore, we have by 7.2.11 $(iv) \implies (i)$, that A is the infinitesimal generator of a contraction strongly continuous semigroup $S(t) : L^2(\Omega) \to L^2(\Omega)$ for ≥ 0. Furthermore, L is self-adjoint by 11.2 (c) with

$$\sup_{0 \neq u \in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)} \frac{\langle u, Lu\rangle}{\|u\|_{L^2(\mathbb{R}^n)}} \leq 0 < \infty,$$

so that S is also self-adjoint by Theorem 7.3.10.

(b) S can be extended into a strongly continuous groups by 7.2.4 exactly if $-L$ is also the infinitesimal generator of a strongly continuous semigroup. The fact of being the infinitesimal generator of a strongly continuous semigroup is characterised by Hille–Yoshida (7.2.8). One necessary condition in (ii) is that the spectrum $\sigma(-L)$ is bounded from above. However, $L^{-1} : L^2(\mathbb{R}^n) \to W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \hookrightarrow L^2(\mathbb{R}^n)$ is compact as the last inclusion in compact by Rellich–Kondrachov and the first operator is bounded. Furthermore, as L is self-adjoint, L^{-1} is self-adjoint as well. By 5.2.16. (v), there is an orthonormal basis of eigenvectors for $L^2(\mathbb{R}^n)$. Furthermore, $\lambda = 0$ is not an eigenvalue, as L is surjective and so by 5.2.7, there are countably many eigenvalues λ_i (every E_{λ_i} has finite dimension whereas $L^2(\mathbb{R}^n)$ is infinite dimensional.) and the eigenvalues have to accumulate at zero. Due to this, $\frac{1}{\lambda_i}$ are eigenvalues of L. This means that all λ_i are negative and so the point spectrum of L is unbounded below, meaning that the spectrum of $-L$ is unbounded from above. Due to our previous observation, this implies that $-L$ is not the infinitesimal generator of a strongly continuous semigroups, and so S cannot be extended to a strongly continuous group.

(c) By 7.4.5, S is an analytical semigroup as every self-adjoint strongly continuous semigroup is. Therefore, $\operatorname{im} S(t) \subset \operatorname{dom}(A)$ for all $t > 0$, furthermore we even have $\operatorname{im} S(t) \subset \operatorname{dom}(A^\infty)$. This is the set defined in 7.1.14. as $\operatorname{dom}(A^\infty) = \bigcap_{n=1}^\infty \operatorname{dom}(A^n)$ with $\operatorname{dom}(A^n) := \{x \in \operatorname{dom}(A) : Ax \in \operatorname{dom}(A^{n-1})\}$.

In our situation, we already established in 11.3 for $L = \Delta$, that $\operatorname{dom}(\Delta^2) = \Gamma$. So in a similar fashion, inductively, we get by elliptic regularity that

$$\operatorname{dom}(L^n) = \{u \in W^{2n,2}(\Omega) \cap W_0^{1,2}(\Omega) : L^i u \in W_0^{1,2}(\Omega) \text{ for } i = 1, \dots, n-1\}.$$

So

$$\operatorname{dom}(L^\infty) \subset (\bigcap_{n=1}^{n} W^{2n,2}(\Omega)) \cap W_0^{1,2}(\Omega) \subset \{u \in C^\infty(\Omega) : u|_{\partial\Omega} = 0\},$$

so all the solutions $x : [0, \infty) \to L^2(\mathbb{R}^n) : t \mapsto S(t)u_0$ is mapping into $C^\infty(\Omega)$ and vanishes on the boundary for $t > 0$ and $u_0 \in L^2(\mathbb{R}^n)$.