13.1. The fundamental solution.

(a) Arrive at the formula for K_t by solving an ODE on the Fourier side for $t > 0$ for solutions u to the heat equation with $u(t, \cdot) \in \mathcal{S}(\mathbb{R}^n)$. **Hint:** The letters x and ξ somehow resemble themselves.

(b) Prove that K_t is the fundamental solution i.e. extend K_t by $K_t \equiv 0$ for $t \leq 0$. Define the distribution $u_K : \mathcal{S}(\mathbb{R}^{n+1}) \to \mathbb{R} : \varphi \to \int_{\mathbb{R}\times\mathbb{R}^n} K(t,x)\varphi(t,x)\, \mathrm{d}t\mathrm{d}x$ and prove that in the distributional sense $Pu_K = \delta_0$ where δ_0 is Dirac's delta distribution $\varphi \to \varphi(0)$.

(c) Prove that there is $C > 0$ such that for every $t > 0$, $\|\nabla_x K_t\|_{L^1(\mathbb{R}^n,\mathbb{R}^n)} \leq \frac{C}{\sqrt{t}}$. Deduce that

$$\|\Delta_x K_t\|_{L^1(\mathbb{R}^n)} \leq \frac{C}{t}, \qquad \left\|\Delta_x^2 K_t\right\| \leq \frac{C}{t^2}.$$

Hint: Use $K_t = K_{t/2} * K_{t/2}$ for the last inequality.

13.2. Gelfand Triples. Let H be a Hilbert space, $V \subset H$ be a dense subspace. Suppose that V is a Hilbert space in its own right with an inner product $\langle \cdot, \cdot \rangle_V$. Identify H with H^* with the canonical isomorphism. Take $\iota : V \to H$ the inclusion map, and $\iota^* : H^* \to V^*$ which are both injective and have dense image by FA I. Under the identifications, $u \in H$ is sent to $V \to \mathbb{R} : v \mapsto \langle u, v \rangle_H$. Thus

$$V \subset H \cong H^* \subset V^*.$$

Take $B : V \times V \to \mathbb{R}$ to be a symmetric bilinear form and suppose there are constants $\delta > 0$, $c > 0$, $C > 0$ such that

$$\delta \|v\|_V^2 - c\|v\|_H^2 \leq B(v,v) \leq C\|v\|_V^2$$

for all $v \in V$. Define $A : \mathrm{dom}(A) \to H$ by

$$\mathrm{dom}(A) := \left\{ u \in V : \sup_{v \in V} \frac{|B(u,v)|}{\|v\|_H} < \infty \right\}, \ \langle Au, v \rangle_H := B(u,v) \text{ for all } v \in V.$$

(a) Prove that A is self-adjoint.

Hint: Follow the hints given at the end of Remark 6.3.8.

(b) Prove that $-A$ generates a strongly continuous semigroup.

Hint: Use Theorem 7.3.10.

(c) Recast the infinitesimal generators of Exercises 12.2 (p=2) and 12.3 in the light of Gelfand triples.

13.3. Maximal principle and exponential decay. Let $T > 0$ and let $\Omega \subset \mathbb{R}^n$ be a bounded, open domain. Set $\Omega_T := (0, T] \times \Omega$ and $\Gamma_T := (\{0\} \times \Omega) \cup ([0, T] \times \partial\Omega)$. Let $Lu = \sum_{i,j} a_{i,j}\partial_i\partial_j u + \sum_{i=1}^n b_i\partial_i u + cu$ for $u \in C^2(\Omega_T)$, $a_{ij} = a_{ji}, b_i, c \in C^0(\overline{\Omega_T})$ and $i, j = 1, \ldots, n$ with

$$\sum_{i,j=1}^n a_{ij}(t,x)\xi_i\xi_j \geq \delta\,|\xi|^2$$

for all $(t, x) \in \overline{\Omega_T}$ and all $\xi \in \mathbb{R}^n$. Put $Pu = Lu - \partial_t u$.

(a) Let $c \leq 0$, $u \in C^2(\Omega_T) \cap C^0(\overline{\Omega_T}), Pu \geq 0$, then prove that

$$\max_{\overline{\Omega_T}} u \leq \max_{\Gamma_T} u^+$$

where $u^+(x) := \max(u(x), 0)$ is the positive part of u.

Hint: Mimic the proof of the maximum principle for $c = 0$ (Theorem 2).

(b) Prove that if there is $\gamma \in \mathbb{R}$ such that $-c \geq \gamma > 0$, then for $g \in L^\infty(\Omega)$ and $u \in C^2(\Omega_T) \cap C^0(\overline{\Omega_T})$ solution of

$$\begin{cases} Pu = 0 \text{ on } & \Omega_T \\ \;\;u = 0 \text{ on } & (0,T) \times \partial\Omega \\ \;\;u = g \text{ on } & \{0\} \times \Omega, \end{cases}$$

we get

$$|u(t,x)| \leq \|g\|_{L^\infty(\mathbb{R}^n)}\, e^{-\gamma t}$$

for all $(t, x) \in \overline{\Omega_T}$.

13.4. Fractional derivatives for $p = 2$. This exercise serves as prelude to the Besov spaces which will appear soon in the lecture.

Define[1]

$$H^s(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n) : (2\pi)^{-n}\int_{\mathbb{R}^n}(1 + |\xi|^2)^s\,|\hat{u}(\xi)|^2 \;\mathrm{d}\xi < \infty\}$$

for all $s \geq 0$. For $u, v \in H^s(\mathbb{R}^n)$, define the scalar product[2]

$$\langle u, v\rangle_s := (2\pi)^{-n}\int_{\mathbb{R}^n}(1 + |\xi|^2)^s\mathcal{F}(u)\overline{\mathcal{F}(v)}\;\mathrm{d}\xi.$$

[1] Recall the Fourier transform and its properties from exercises 8.4 and 8.5.
[2] As always we use both the hat notation and $\mathcal{F}$ to denote the Fourier transform on $L^2(\mathbb{R}^n)$.

(a) Prove $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$. Prove that H^s is a Hilbert space.

Hint: For the second statement use completeness of $L^2((1+|\xi|^2)^s \, \mathrm{d}\xi) \subset L^2(\mathrm{d}\xi)$.

(b) Prove that $W^{k,2}(\mathbb{R}^n) = H^k(\mathbb{R}^n)$ for $k \in \mathbb{N}$.

Hint: Start with $k = 1$ to test the ground. Prove the equivalence of the norms on $\mathcal{S}(\mathbb{R}^n)$.

(c) Prove that for $2s > n$, $H^s(\mathbb{R}^n)$ imbeds continuously into $C^0(\mathbb{R}^n)$.

Hint: Use the Fourier inverse formula.

Please hand in your solutions for this sheet by Monday 30/05/2016. Your teaching assistants will put your corrected sheets in their pigeonholes in F28.