13.1. The fundamental solution.

(a) Arrive at the formula for K_t by solving an ODE on the Fourier side for $t > 0$ for solutions u to the heat equation with $u(t, \cdot) \in \mathcal{S}(\mathbb{R}^n)$. **Hint:** The letters x and ξ somehow resemble themselves.

(b) Prove that K_t is the fundamental solution i.e. extend K_t by $K_t \equiv 0$ for $t \leq 0$. Define the distribution $u_K : \mathcal{S}(\mathbb{R}^{n+1}) \to \mathbb{R} : \varphi \to \int_{\mathbb{R}\times\mathbb{R}^n} K(t, x)\varphi(t, x)\, \mathrm{d}t\mathrm{d}x$ and prove that in the distributional sense $Pu_K = \delta_0$ where δ_0 is Dirac's delta distribution $\varphi \to \varphi(0)$.

(c) Prove that there is $C > 0$ such that for every $t > 0$, $\|\nabla_x K_t\|_{L^1(\mathbb{R}^n,\mathbb{R}^n)} \leq \frac{C}{\sqrt{t}}$. Deduce that

$$\|\Delta_x K_t\|_{L^1(\mathbb{R}^n)} \leq \frac{C}{t}, \qquad \left\|\Delta_x^2 K_t\right\| \leq \frac{C}{t^2}.$$

Hint: Use $K_t = K_{t/2} * K_{t/2}$ for the last inequality.

Solution:

(a) Assume for $t \geq 0$, $u(t, \cdot) \in \mathcal{S}(\mathbb{R}^n)$ is solution to the heat equation $\partial_t u = \Delta u$ with $u(0, \cdot) = u_0$. Then taking the Fourier transform of this equation with respect to x, we get with local notation $\mathcal{G}(u)(t, \xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} u(t, x)\, \mathrm{d}x$,

$$\partial_t \mathcal{G}(u)(t, \xi) = \mathcal{G}(\partial_t u)(t, \xi) = \mathcal{G}(\Delta u)(t, \xi) = -|\xi|^2 \mathcal{G}(u)(t, \xi)$$

for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$. Thus for every $\xi \in \mathbb{R}^n$, we solve the ODE in t to get

$$\mathcal{G}(u)(t, \xi) = e^{-|\xi|^2 t}\mathcal{F}(u_0)(\xi)$$

for all $\xi \in \mathbb{R}^n$ and $t \geq 0$. Now if we apply the inverse Fourier transform in ξ to this formula for $t > 0$, then we find

$$u(t, x) = (K_t * u_0)(x)$$

for all $t > 0$ and $x \in \mathbb{R}^n$, because of products going over to convolution products under Fourier transforms and $\hat{K}_t(\xi) = e^{-|\xi|^2 t}$ as can be found in the solution of exercise 12.2.

(b) Let $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$. We want to prove that K_t extended by zero for $t \leq 0$, fulfils

$$I := \int_{\mathbb{R}^{n+1}} K(t, x)(-\partial_t\varphi(t, x) - \Delta\varphi(t, x))\, \mathrm{d}x\mathrm{d}t = \varphi(0, 0).$$

So we calculate

$$
\begin{aligned}
I &= \lim_{\delta\to 0^+}\int_\delta^\infty \int_{\mathbb{R}^n} K_t(t,x)(-\partial_t\varphi(t,x)-\Delta\varphi(t,x))\ \mathrm{d}x\mathrm{d}t\\
&= \lim_{\delta\to 0^+}\int_\delta^\infty \int_{\mathbb{R}^n} P(K_t(t,x))\varphi\ \mathrm{d}x\mathrm{d}t + \lim_{\delta\to 0^+}\int_{\mathbb{R}^n} K_\delta(x)\varphi(\delta,x)\ \mathrm{d}x\\
&= 0 + \lim_{\delta\to 0^+}\int_{\mathbb{R}^n} K_\delta(x)(\varphi(\delta,x)-\varphi(0,0))\ \mathrm{d}x + \varphi(0,0)\\
&= \varphi(0,0)
\end{aligned}
$$

where the first line uses dominated convergence, the second one uses integration by part in the space variables (no boundary terms due to compact support.) and the time variable (only one boundary term, the other one vanishes.). In the third line, we use $PK_t = 0$ for $t > 0$ and $\int_{\mathbb{R}^n} K_t = 1$ for $t > 0$. What remains to be proven is that

$$
\lim_{\delta\to 0^+}\int_{\mathbb{R}^n} K_\delta(x)(\varphi(\delta,x)-\varphi(0,0))\ \mathrm{d}x = 0
$$

We use the fact proven in class that $\lim_{\delta\to 0^+}\int_{|x|\geq\eta} K_\epsilon(x)\ \mathrm{d}x = 0$ for all $\eta > 0$. Fix $\epsilon > 0$. Then, there is $\rho > 0$ such that for $|(t,x)-(0,0)| \leq \rho$, then $|\varphi(t,x)-\varphi(0,0)| < \frac{\epsilon}{2}$. Now for $\eta = \frac{\rho}{2}$ there is $\alpha > 0$ such that for all $0 < \delta < \alpha$,

$$
\int_{|x|\geq\eta} K_\epsilon(x)\ \mathrm{d}x < \frac{\epsilon}{4\left\|\varphi\right\|_{L^\infty(\mathbb{R}^n)}}.
$$

Now for $\delta \leq \min(\alpha, \frac{\rho}{2})$, we have

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^n} K_\delta(x)(\varphi(\delta,x)-\varphi(0,0))\ \mathrm{d}x\right|\\
&\qquad\leq 2\left\|\varphi\right\|_{L^\infty(\mathbb{R}^n)}\int_{|x|\geq\frac{\rho}{2}} K_\delta(x)\ \mathrm{d}x + \sup_{|x|\leq\frac{\rho}{2}}|\varphi(x,\delta)-\varphi(0,0)|\int_{B_{\frac{\rho}{2}}(0)} K_\delta(x)\ \mathrm{d}x\\
&\qquad< \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.
\end{aligned}
$$

This proves that K_t is the fundamental solution of P.

(c) $\nabla_x K_t(x) = \frac{x}{2t}K_t(x)$, so by rescaling $\|\nabla_x K_t(x)\| = \frac{1}{\sqrt{t}}\frac{2}{\pi^{n/2}}\int_{\mathbb{R}^n} e^{-|x|^2}|x| = \frac{C}{\sqrt{t}}$.

$\Delta_x K_t(x) = (\frac{|x|^2}{4t^2}-\frac{n}{2t})K_t(x)$, and so again by rescaling, we have $\|\Delta_x K_t\|_{L^1(\mathbb{R}^n)} \leq \frac{C}{t}$.

Now from $\Delta_x^2 K_t = \partial_t^2 K_t$, we can write $K_t = K_{t/2} * K_{t/2}$ and so

$$
\Delta_x^2 K_t = \partial_t^2 K_t = \frac{1}{4}(\partial_t K_{t/2}) * (\partial_t K_{t/2}) = \frac{1}{4}\Delta_x K_{t/2} * \Delta_x K_{t/2}.
$$

By Young's inequality we have

$$
\left\|\Delta_x^2 K_t\right\|_{L^1(\mathbb{R}^n)} = \frac{1}{4}\left\|\Delta_x K_{t/2}\right\|_{L^1(\mathbb{R}^n)}^2 \leq \frac{C^2}{t^2}.
$$

13.2. Gelfand Triples. Let H be a Hilbert space, $V \subset H$ be a dense subspace. Suppose that V is a Hilbert space in its own right with an inner product $\langle \cdot, \cdot \rangle_V$. Identify H with H^* with the canonical isomorphism. Take $\iota : V \to H$ the inclusion map, and $\iota^* : H^* \to V^*$ which are both injective and have dense image by FA I. Under the identifications, $u \in H$ is sent to $V \to \mathbb{R} : v \mapsto \langle u, v \rangle_H$. Thus

$$V \subset H \cong H^* \subset V^*.$$

Take $B : V \times V \to \mathbb{R}$ to be a symmetric bilinear form and suppose there are constants $\delta > 0$, $c > 0$, $C > 0$ such that

$$\delta \|v\|_V^2 - c \|v\|_H^2 \leq B(v, v) \leq C \|v\|_V^2$$

for all $v \in V$. Define $A : \mathrm{dom}(A) \to H$ by

$$\mathrm{dom}(A) := \left\{ u \in V : \sup_{v \in V} \frac{|B(u, v)|}{\|v\|_H} < \infty \right\}, \ \langle Au, v \rangle_H := B(u, v) \text{ for all } v \in V.$$

(a) Prove that A is self-adjoint.

Hint: Follow the hints given at the end of Remark 6.3.8.

(b) Prove that $-A$ generates a strongly continuous semigroup.

Hint: Use Theorem 7.3.10.

(c) Recast the infinitesimal generators of Exercises 12.2 (p=2) and 12.3 in the light of Gelfand triples.

Solution:

(a) That A is symmetric comes from B being symmetric. We start by proving that $A + c\mathbb{1} : \mathrm{dom}(A) \to H$ is bijective. We observe that for all $x \in V$, we have

$$\delta \|x\|_V^2 \leq B(x, x) + c \|x\|_H^2 \leq C \|x\|_V^2 + c \|x\|_H^2 \leq (C + C'c) \|x\|_V^2 .$$

where C' is the norm of ι. Thus $V \times V \to \mathbb{R} : (u, v) \mapsto c \langle u, v \rangle_H + B(u, v)$ is a scalar product equivalent to $\langle \cdot, \cdot \rangle_V$ on V. So it makes V also into a Hilbert space. Now by Riesz representation theorem for Hilbert spaces, we have for $f \in H$, which defines a linear functional on V by $v \mapsto \langle v, f \rangle_H$, that there is $u \in V$ such that

$$c \langle u, v \rangle_H + B(u, v) = \langle f, v \rangle_H$$

for all $v \in V$. By Cauchy-Schwarz, we get that

$$\sup_{0 \neq u \in V} \frac{|B(u, v)|}{\|v\|_H} \leq \|f\|_H + C' \|v\|_H < \infty.$$

Hence $v \in \operatorname{dom}(A)$ and $cu + Au = f$ by density of V in H. So in particular, we have that $A + c\mathbb{1} : \operatorname{dom}(A) \to H$ is surjective. Also for all $u \in \operatorname{dom}(A)$, we have $Au + cu = 0$ that,

$$\delta \|u\|_V^2 \leq B(u,u) + c\|u\|_H^2 = \langle Au + cu, u\rangle_H = 0.$$

Thus this norm is zero so $u = 0$.

Now as for every $v^* \in V^*$, there is a unique $u \in V$ such that $B(u,\cdot) + c\langle u, \cdot\rangle_H = v^*(\cdot)$. This extends the map $c\mathbb{1} + A : \operatorname{dom}(A) \subset V \to H \subset V^*$ into an isomorphism from $V \to V^*$ which sends $\operatorname{dom}(A)$ to H. Now as $H \subset V^*$ is dense, so is $\operatorname{dom}(A)$ in V.

Next, let us prove that for $u \in H$, if there is $c > 0$ such that for all $v \in \operatorname{dom}(A)$, we have

$$|\langle u, Av\rangle_H| \leq c\sqrt{B(v,v) + c\|v\|_H^2},$$

then $u \in V$.

To prove this, we use density of $\operatorname{dom}(A)$ in V to extend the map $v \mapsto \langle u, Av\rangle_H + c\langle u, v\rangle_H$ uniquely to a map $v^* \in V^*$. Then, there is as before, $u' \in V$ such that

$$B(u', v) + c\langle u', v\rangle = v^*(v)$$

for all $v \in V$. Thereby, for $v \in \operatorname{dom}(A)$, we obtain

$$\langle u', Av\rangle_H + c\langle u', v\rangle_H = B(u', v) + c\langle u', v\rangle = \langle u, Av\rangle_H + c\langle u, v\rangle_H$$

by definition of A. As $A + c\mathbb{1} : \operatorname{dom}(A) \to H$ is surjective, we have that $u = u' \in V$.

Thus assume that $u, w \in H$ such that for all $v \in \operatorname{dom}(A)$, we have

$$\langle u, Av\rangle_H = \langle w, v\rangle_H.$$

By the previous argument, as we have

$$|\langle u, Av\rangle_H| \leq \|w\|_H \|v\|_H \leq \|w\|_H \sqrt{B(v,v) + c\|v\|_H^2},$$

we get $u \in V$. Thus, $B(u,v) = \langle w, v\rangle_H$ for all $v \in \operatorname{dom}(A)$, which, by density of $\operatorname{dom}(A)$ in V, continues to hold for all $v \in V$. This exactly means that $u \in \operatorname{dom}(A)$ and that $Au = w$. Thus proving that A is self-adjoint.

(b) By the previous point, $-A : \operatorname{dom}(A) \to H$ is self-adjoint.

$$\langle -Av, v\rangle = -B(v,v) \leq c\|u\|_H - \delta\|u\|_V \leq c\|u\|_H^2.$$

Hence the hypotheses of 7.3.10 are met and so $-A$ is infinitesimal generator of a strongly continuous self-adjoint semigroup.

(c) For exercise 12.2., we take $H = L^2(\mathbb{R}^n)$, $V = W^{1,2}(\mathbb{R}^n)$ and $B(u,v) = \langle \nabla u, \nabla v \rangle_{L^2(\mathbb{R}^n)}$ for all $(u,v) \in V \times V$. Then we get the lower bound $B(u,u) = \|\nabla u\|^2_{L^2(\mathbb{R}^n)} = \|u\|^2_{W^{1,2}(\mathbb{R}^n)} - \|u\|^2_{L^2(\mathbb{R}^n)}$, so $c = \delta = 1$. Furthermore, we have $B(u,u) \leq \|u\|_{W^{1,2}(\mathbb{R}^n)}$. So $C = 1$, and by elliptic regularity, we get $A = -\Delta : W^{2,2}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

For exercise 12.3, we take $V = W^{1,2}_0(\Omega)$, $H = L^2(\Omega)$, $B(u,v) := \int_\Omega \partial_i u \; a_{ij} \; \partial_j u$ for all $(u,v) \in V \times V$, then we can take $c = 0$. The lower bound is given by ellipticity of the a_{ij} and the upper bound by boundedness of the coefficients. By elliptic regularity, we have $A = -L : W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \to L^2(\Omega)$.

13.3. Maximal principle and exponential decay. Let $T > 0$ and let $\Omega \subset \mathbb{R}^n$ be a bounded, open domain. Set $\Omega_T := (0,T] \times \Omega$ and $\Gamma_T := (\{0\} \times \Omega) \cup ([0,T] \times \partial\Omega)$. Let $Lu = \sum_{i,j} a_{i,j} \partial_i \partial_j u + \sum_{i=1}^n b_i \partial_i u + cu$ for $u \in C^2(\Omega_T)$, $a_{ij} = a_{ji}, b_i, c \in C^0(\overline{\Omega_T})$ and $i, j = 1, \dots, n$ with

$$\sum_{i,j=1}^n a_{ij}(t,x) \xi_i \xi_j \geq \delta \left|\xi\right|^2$$

for all $(t,x) \in \overline{\Omega_T}$ and all $\xi \in \mathbb{R}^n$. Put $Pu = Lu - \partial_t u$.

(a) Let $c \leq 0$, $u \in C^2(\Omega_T) \cap C^0(\overline{\Omega_T})$, $Pu \geq 0$, then prove that

$$\max_{\overline{\Omega_T}} u \leq \max_{\Gamma_T} u^+$$

where $u^+(x) := \max(u(x), 0)$ is the positive part of u.

Hint: Mimic the proof of the maximum principle for $c = 0$ (Theorem 2).

(b) Prove that if there is $\gamma \in \mathbb{R}$ such that $-c \geq \gamma > 0$, then for $g \in L^\infty(\Omega)$ and $u \in C^2(\Omega_T) \cap C^0(\overline{\Omega_T})$ solution of

$$\begin{cases} Pu = 0 \text{ on } & \Omega_T \\ u = 0 \text{ on } & (0,T) \times \partial\Omega \\ u = g \text{ on } & \{0\} \times \Omega, \end{cases}$$

we get

$$|u(t,x)| \leq \|g\|_{L^\infty(\mathbb{R}^n)} e^{-\gamma t}$$

for all $(t,x) \in \overline{\Omega_T}$.

Solution:

(a) We simply repeat the proof of Theorem 2 (maximum principle for parabolic equations) word for word.

We assume $M := \max_{\overline{\Omega_T}} u > \max_{\Gamma_T} u^+ \geq 0$. Choose $(t_0, x_0) \in \Omega_T$ with $u(t_0, x_0) = M$. Let $\theta > 0$ such that

$$v(t,x) = e^{-\theta(t-t_0)-|x-x_0|^2}$$

has the property

$$\begin{aligned} Pv(t,x) = \Bigg[4\sum_{i,j=1}^{n} a_{ij}(t,x)(x_i-(x_0)_i)(x_j-(x_0)_j) - 2\sum_{i=1}^{n} a_{ii}(t,x) \\ -2\sum_{i=1}^{n} b_i(t,x)(x_i-(x_0)_i) + c(t,x) + \theta \Bigg] e^{-\theta(t-t_0)-|x-x_0|^2} > 0 \end{aligned}$$

for all $(t,x) \in \Omega_T$ which is possible as Ω and $T > 0$ are bounded. Next choose $\epsilon > 0$ such that

$$\epsilon(e^{\theta T} - 1) < M - \sup_{\Gamma_T} u^+.$$

Define $u_\epsilon := u + \epsilon v$. Then we still have

$$\max_{\Gamma_T} u_\epsilon^+ \leq \max_{\Gamma_T} u^+ + \epsilon \max_{\Gamma_T} v_\epsilon \leq \max_{\Gamma_T} u + \epsilon e^{\theta T} < M + \epsilon = u(t_0,x_0) + \epsilon v(t_0,x_0) \leq \max_{\overline{\Omega_T}} u_\epsilon,$$

but now we also have

$$Pu_\epsilon = Pu + \epsilon Pv > 0.$$

Therefore, choose $(t_1, x_1) \in \overline{\Omega}_T$ with $u_\epsilon(t_1,x_1) = \max_{\overline{\Omega_T}} u_\epsilon > \max_{\Gamma_T} u_\epsilon \geq 0$. Then, we have

$$\begin{aligned} Pu_\epsilon(t_1,x_1) = \sum_{i,j=1}^{n} a_{ij}(t_1,x_1)\partial_i\partial_j u_\epsilon(t_1,x_1) + \sum_{i=1}^{n} b_i(t_1,x_1)\partial_i u_\epsilon(t_1,x_1) \\ + c(t_1,x_1)u_\epsilon(t_1,x_1) - \partial_t u_\epsilon(t_1,x_1) \leq 0 \end{aligned}$$

where every single summands is ≤ 0 due to u_ϵ attaining a maximum and $c \leq 0$. This provides us with a contradiction.

(b) We look at $v(t,x) = u(t,x) - \|g\|_{L^\infty(\Omega)} e^{-\gamma t}$ for $(t,x) \in \overline{\Omega}_T$. Then we see that as $u|_{(0,\infty)\times\partial\Omega} = 0$, that

$$\max_{\Gamma_T} v \leq 0.$$

Furthermore, we have that

$$Pv(t,x) = 0 - c\,\|g\|_{L^\infty(\Omega)}\,e^{-\gamma t} - \gamma\,\|g\|_{L^\infty(\Omega)}\,e^{-\gamma t} \geq 0.$$

for $(t,x) \in \overline{\Omega}_T$. Thus, we may apply (a), to conclude that

$$\max_{\overline{\Omega}_T} v \leq 0.$$

This translates into

$$u(t,x) \leq \|g\|_{L^\infty(\Omega)}\,e^{-\gamma t}.$$

for $(t,x) \in \overline{\Omega}_T$. The same argument with $\tilde{v}(t,x) = -u(t,x) - \|g\|_{L^\infty(\Omega)}\,e^{-\gamma t}$ delivers the second inequality

$$\|g\|_{L^\infty(\Omega)}\,e^{-\gamma t} \leq u(t,x)$$

for $(t,x) \in \overline{\Omega}_T$.

13.4. Fractional derivatives for $p = 2$. This exercise serves as prelude to the Besov spaces which will appear soon in the lecture.

Define[1]

$$H^s(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n) : (2\pi)^{-n} \int_{\mathbb{R}^n} (1 + |\xi|^2)^s\,|\hat{u}(\xi)|^2 \,\mathrm{d}\xi < \infty\}$$

for all $s \geq 0$. For $u, v \in H^s(\mathbb{R}^n)$, define the scalar product[2]

$$\langle u, v\rangle_s := (2\pi)^{-n} \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \mathcal{F}(u)\overline{\mathcal{F}(v)}\,\mathrm{d}\xi.$$

(a) Prove $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$. Prove that H^s is a Hilbert space.

Hint: For the second statement use completeness of $L^2((1 + |\xi|^2)^s\,\mathrm{d}\xi) \subset L^2(\mathrm{d}\xi)$.

(b) Prove that $W^{k,2}(\mathbb{R}^n) = H^k(\mathbb{R}^n)$ for $k \in \mathbb{N}$.

Hint: Start with $k = 1$ to test the ground. Prove the equivalence of the norms on $\mathcal{S}(\mathbb{R}^n)$.

(c) Prove that for $2s > n$, $H^s(\mathbb{R}^n)$ imbeds continuously into $C^0(\mathbb{R}^n)$.

Hint: Use the Fourier inverse formula.

[1] Recall the Fourier transform and its properties from exercises 8.4 and 8.5.

[2] As always we use both the hat notation and $\mathcal{F}$ to denote the Fourier transform on $L^2(\mathbb{R}^n)$.

Solution:

(a) The first statement is exactly Plancherel's identity which states

$$\|u\|_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \|\hat{u}\|_{L^2(\mathbb{R}^n)} .$$

For the second statement, it is not out of this world to check that $H^s(\mathbb{R}^n)$ is a linear subspace and that $\langle \cdot, \cdot \rangle_s$ defines a scalar product using that $\mathcal{F}$ is a linear isomorphism. To prove that $H^s(\mathbb{R}^n)$ is complete, we take a Cauchy sequence u_i in $H^s(\mathbb{R}^n)$, then $\mathcal{F}(u_i)$ is a Cauchy sequence in $L^2((1+|\xi|^2)^s \,\mathrm{d}\xi)$ where $(1+|\xi|^2)^s \,\mathrm{d}\xi$ is the absolute continuous measure with respect to the Lebesgue measure on $\mathbb{R}^n$ with Radon–Nikodym derivative $(1+|\xi|^2)^s$. This space is complete, so there is $v \in L^2((1+|\xi|^2)^s \mathbb{R}^n)$ such that $\mathcal{F}u_i$ converges to v in $L^2((1+|\xi|^2)^s \,\mathrm{d}\xi)$. As $L^2((1+|\xi|^2)^s \,\mathrm{d}\xi) \subset L^2(\mathrm{d}\xi)$, there is $u \in L^2(\mathbb{R}^n)$ such that $\mathcal{F}u = v$. Now it follows directly that $u_i \to u$ in $H^s(\mathbb{R}^n)$. So $H^2(\mathbb{R}^n)$ is indeed a Hilbert space.

(b) Fix $s = k \in \mathbb{N}$. Then for $u \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\begin{aligned}
\|u\|^2_{W^{k,2}(\mathbb{R}^n)} &= \sum_{|\alpha| \leq k} \|\partial^\alpha u\|^2_{L^2(\mathbb{R}^n)} = (2\pi)^{-n} \sum_{|\alpha| \leq k} \|\mathcal{F}(\partial^\alpha u)\|^2_{L^2(\mathbb{R}^n)} \\
&= (2\pi)^{-n} \sum_{|\alpha| \leq k} \left\| i^{|\alpha|} \xi^\alpha \hat{u} \right\|_{L^2(\mathbb{R}^n)} = (2\pi)^{-n} \int_{\mathbb{R}^n} \big(\sum_{|\alpha| \leq k} |\xi|^{2\alpha} \big) \left|\hat{u}(\xi)\right|^2 \,\mathrm{d}\xi
\end{aligned}$$

Now for $\xi \in \mathbb{R}^n$, we have

$$(1+|\xi|^2)^k = (1+\xi_1^2+\xi_2^2+\ldots+\xi_n^2)^k = \sum_{|\alpha| \leq k} \binom{n+1}{(k-|\alpha|, \alpha)} |\xi|^{2\alpha}$$

where we used the multinomial formula for $x = (1, \xi_1^2, \ldots, \xi_n^2) \in \mathbb{R}^{n+1}$. As each coefficient is at least 1, we get directly

$$\sum_{|\alpha| \leq k} |\xi|^{2\alpha} \leq (1+|\xi|^2)^k \leq C \sum_{|\alpha| \leq k} |\xi|^{2\alpha}$$

for some $C := C(k, m) > 0$ for all $\xi \in \mathbb{R}^n$. Thus, we get

$$\|u\|^2_{W^{k,2}(\mathbb{R}^n)} \leq \|u\|^2_{H^s(\mathbb{R}^n)} \leq C \|u\|^2_{W^{k,2}(\mathbb{R}^n)}$$

We already have that $\mathcal{S}(\mathbb{R}^n)$ is dense in $W^{k,2}(\mathbb{R}^n)$. Let us show that $\mathcal{S}(\mathbb{R}^n)$ is also dense in $H^s(\mathbb{R}^n)$. Let $u \in H^s(\mathbb{R}^n)$ and approximate $\hat{u}(1+|\xi|^2)^{s/2} \in L^2(\mathbb{R}^n)$ by $u_i \in \mathcal{S}(\mathbb{R}^n)$ in L^2, then taking $v_i := \mathcal{F}^{-1}\left(\frac{u_i}{(1+|\xi|^2)^{s/2}}\right) \in \mathcal{S}(\mathbb{R}^n)$. Then v_i converges to u in $H^s(\mathbb{R}^n)$.

So the completions of $\mathcal{S}(\mathbb{R}^n)$ agrees with respect to both norms, so $H^k(\mathbb{R}^n) = W^{k,2}(\mathbb{R}^n)$.

(c) We want to show that $u \in H^s(\mathbb{R}^n)$ admits a continuous representative. So fix $u \in H^s(\mathbb{R}^n)$ and approximate by Schwartz functions $u_i \in \mathcal{S}(\mathbb{R}^n)$ which is possible by the same argument as above. Then we want to prove that u_i is also a Cauchy sequence with respect to the sup-norm. By Fourier inverse formula, we have

$$u_i(x) - u_j(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} (\hat{u}_j(\xi) - \hat{u}_i(\xi)) \, \mathrm{d}\xi$$

Therefore, we get

$$\begin{aligned}
|u_i(x) - u_j(x)| &\leq (2\pi)^{-n} \int_{\mathbb{R}^n} |\hat{u}_j(\xi) - \hat{u}_i(\xi)| \, \mathrm{d}\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} (1 + |\xi|^2)^{s/2} |\hat{u}_j(\xi) - \hat{u}_i(\xi)| \frac{1}{(1 + |\xi|^2)^{s/2}} \, \mathrm{d}\xi \\
&\leq \|u_j - u_i\|_s \int_{\mathbb{R}^n} \frac{1}{(1 + |\xi|^2)^s} \mathrm{d}\xi \leq C \|u_i - u_j\|_s
\end{aligned}$$

where we used Hölder with $p = q = 2$ in the penultimate inequality and for the last inequality, we use $\frac{1}{(1+|\xi|^2)^s} \leq \frac{1}{(1+|\xi|^{2s})} \in L^1(\mathbb{R}^n)$ due to $2s > n$. Thus u_i is Cauchy in sup-norm on $\mathbb{R}^n$ as C is independent of x. Therefore, there is $\tilde{u} \in C^0(\mathbb{R}^n)$ such that $u_i \to \tilde{u}$ in sup-norm. By the same argument, we get that u_j also point-wise converges almost everywhere to u. So $u = \tilde{u}$ almost everywhere (the Fourier inverse formula is only true almost everywhere for L^2 functions.). Furthermore for two $u, v \in H^s(\mathbb{R}^n)$, we get $\sup_{\mathbb{R}^n}(\tilde{u} - \tilde{v}) \leq C \|u - v\|_s$. Therefore, we see that $H^s(\mathbb{R}^n) \to C^0(\mathbb{R}^n) : u \to \tilde{u}$ is bounded and injective.