14.1. $L^q - L^p$ spaces. Let $1 < p, q < \infty$.

(a) Define

$$A_{q,p} := \{u : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R} : u \text{ is measurable, } \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |u(t,x)|^p \, \mathrm{d}x\right)^{\frac{q}{p}} \, \mathrm{d}t\right)^{\frac{1}{q}} < \infty\}/_\sim.$$

where $\sim$ is equivalence almost everywhere. Prove that $A_{q,p} \cong L^q(\mathbb{R}, L^p(\mathbb{R}^n))$.

Hint: Use strong measurability to get a sequence of step functions $u_i \in L^q(\mathbb{R}, L^p(\mathbb{R}^n))$ and establish the correspondance. Use $v_k = \sum_{i=0}^k |u_{i+1} - u_i|$ with

$$\|u_{i+1} - u_i\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq \frac{1}{2^i}$$

and the norm on $L^1([-T,T], L^1(K))$ for $T > 0$ and $K \subset \mathbb{R}^n$ compact to prove that the limiting function is measurable.

(b) Prove that $C_0^\infty(\mathbb{R} \times \mathbb{R}^n) \subset L^q(\mathbb{R}, L^p(\mathbb{R}^n))$ is dense.

Hint: Reduce it to step functions and then smoothen every single value, and then smoothen in the t-variable.

(c) Let ρ_δ be a family of mollifiers on $\mathbb{R}^n$ and α_δ a family of mollifiers on $\mathbb{R}$. We introduce $T_\delta : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n) : f \to \rho_\delta * f$ and define for $u \in L^q(\mathbb{R}, L^p(\mathbb{R}^n))$, $S_\delta u : \mathbb{R} \to L^p(\mathbb{R}^n) : t \to T_\delta(u(t))$. Prove that $S_\delta u \in L^q(\mathbb{R}, L^p(\mathbb{R}^n))$ and that $\alpha_\delta * (S_\delta u) \to u$ in $L^q(\mathbb{R}, L^p(\mathbb{R}^n))$.

Hint: Banach–Steinhaus (2.1.5).

(d) Prove that $C_0^\infty(\mathbb{R} \times \mathbb{R}^n)$ is dense in $W^{1,q}(\mathbb{R}, L^p(\mathbb{R}^n)) \cap L^q(\mathbb{R}, W^{2,p}(\mathbb{R}^n))$.

Solution: Let μ be the Lebesgue measure.

(a) Let $s \in L^q(\mathbb{R}, L^p(\mathbb{R}^n))$ for $1 < p, q < \infty$ be a step function. Then

$$s = \sum_{i=1}^N \mathbb{1}_{I_i}[f_i]_\mu$$

where $I_i \subset \mathbb{R}$ are measurable and form a partition a subset of $\mathbb{R}$ and $[f_i]_\mu$ are finitely many equivalence classes of $L^p(\mathbb{R}^n)$ functions. We can choose representatives f_i for these finitely many $L^p(\mathbb{R}^n)$ functions, and define $\tilde{s} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde{s}(t,x) = \sum_{i=1}^N \mathbb{1}_{I_i}(t) f_i(x),$$

which is measurable and any such choice of representatives agrees with $\tilde{s}$ on a set of measure zero. The norm is also equal as

$$\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}^n}|\tilde{s}(t,x)|^p\ \mathrm{d}x\right)^{p/q}\ \mathrm{d}t\right)^{1/q}=\left(\sum_{i=1}^N\mu(I_i)\,\|f_i\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q}=\|s\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))}.$$

Now by strong measurability, we have that step function u_i which converge u in $L^q(\mathbb{R},L^p(\mathbb{R}^n))$ for $i\in\mathbb{N}$. Put $u_0=0$ and take a subsequence (still denoted u_i) such that $i\in\mathbb{N}$,

$$\|u_{i+1}-u_i\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))}\le 2^{-i}.$$

Now define

$$v_n=\sum_{i=0}^n|\tilde{u}_{i+1}-\tilde{u}_i|,\qquad v=\sum_{i=0}^\infty|\tilde{u}_{i+1}-\tilde{u}_i|,$$

and take $K\subset\mathbb{R}^n$ compact and $[-T,T]\subset\mathbb{R}$ for $T>0$. Then we have

$$\begin{aligned}\|v_n\|_{L^1([-T,T]\times K)}&\le\sum_{i=0}^n\int_{-T}^T\int_K|\tilde{u}_{i+1}(t,x)-\tilde{u}_i(t,x)|\ \mathrm{d}x\ \mathrm{d}t\\&\le\sum_{i=0}^n\int_{-T}^T\mu(K)^{1/p^*}\,\|\tilde{u}_{i+1}(t,\cdot)-\tilde{u}_i(t,\cdot)\|_{L^p(K)}\ \mathrm{d}t\\&\le\mu(K)^{1/p^*}(2T)^{1/q^*}\sum_{i=0}^n\|u_{i+1}-u_i\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))}\\&\le\mu(K)^{1/p^*}(2T)^{1/q^*}(\|u_1\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))}+1)\end{aligned}$$

where $\frac{1}{r}+\frac{1}{r^*}=1$ for $r=p,q$. Hence the $L^1([-T,T]\times K)$ norm of v is finite. So, v is measurable and v is finite almost everywhere on $\mathbb{R}\times\mathbb{R}^n$ by σ-compactness. But this means that $\tilde{u}_i$ converges pointwise almost everywhere to a measurable function $\tilde{u}$. This function has also the same norm, i.e.

$$\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))}=\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}^n}|\tilde{u}(t,x)|^p\ \mathrm{d}x\right)^{\frac{q}{p}}\ \mathrm{d}t\right)^{\frac{1}{q}}$$

Hence $u\mapsto[\tilde{u}]_\sim$ is an isomorphism which preserves the norm.

(b) Let us start with $u\in L^q(\mathbb{R},L^p(\mathbb{R}^n))$, we want to prove that this can be approximated by functions of the subspace $C_0^\infty(\mathbb{R}\times\mathbb{R}^n)$. We can approximate u by a step function s. We can approximate now every $f_i\in L^p(\mathbb{R}^n)$ by $g_i\in C_0^\infty(\mathbb{R}^n)$ with $\|f_i-g_i\|_{L^p(\mathbb{R}^n)}\le\frac{\epsilon}{N\mu(I_i)^{1/q}}$. Then

$$\left\|s-\sum_{i=1}^N\mathbb{1}_{I_i}[g_i]_\mu\right\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))}\le\left(\sum_{i=1}^N\mu(I_i)\frac{\epsilon^q}{N\mu(I_i)}\right)^{1/q}=\epsilon.$$

So we may assume that we choose smooth values for our step function.

We may choose sets for $i = 1, \dots, N$, with

$$K_i \subset I_i \subset U_i, \qquad \mu(U_i \setminus K_i) \leq \frac{\epsilon}{N \|g_i\|^q_{L^p(\mathbb{R}^n)}}$$

for K_i compact and U_i open. Next we may by Urysohn's lemma choose

$$\psi_i : \mathbb{R} \to [0,1], \qquad \psi_i \equiv 1 \text{ on } K_i, \qquad \operatorname{supp}(\psi_i) \subset U_i, \qquad \psi_i \in C^0$$

Furthermore, take $T > 0$ such that

$$\sum_{i=1}^N \mu(I_i \setminus [-T,T]) \|g_i\|^q_{L^p(\mathbb{R}^n)} < \epsilon$$

and choose

$$\psi_T : \mathbb{R} \to [0,1], \quad \psi_T \equiv 1 \text{ on } [-T,T], \quad \operatorname{supp}(\psi_T) \subset [-T-1, T+1], \quad \psi_T \in C^0.$$

Then we may define $\hat{s} := \psi_T \sum_{i=1}^N \psi_i g_i$ and see that $|\mathbb{1}_{I_i} - \psi_T \psi_i| \leq \mathbb{1}_{U_i \setminus K_i} + \mathbb{1}_{I_i \setminus [-T,T]}$.

$$\int_{\mathbb{R}} \|s(t,\cdot) - \hat{s}(t,\cdot)\|^q_{L^p(\mathbb{R}^n)} \, \mathrm{d}t \leq \sum_{i=1}^N \int_{U_i \setminus K_i} \|g_i\|^q_{L^p(\mathbb{R}^n)} \, \mathrm{d}t + \sum_{i=1}^N \int_{I_i \setminus [-T,T]} \|g_i\|^q_{L^p(\mathbb{R}^n)} \, \mathrm{d}t \leq 2\epsilon$$

So we can approximate s by $\hat{s} \in C_c(\mathbb{R} \times \mathbb{R}^n)$ such that $\hat{s}$ is smooth in the n last variables.

Now the standard mollifier α_δ argument will work, i.e. we approximate $\hat{s}$ by

$$s_\delta(t,x) := \int_{\mathbb{R}} \alpha_\delta(t-s) \hat{s}(s,x) \, \mathrm{d}s$$

and the usual argument shows that s_δ converges uniformly to $\hat{s}$ as $\delta \to 0$, so in particular, it will converge to $\hat{s}$ in $L^q(\mathbb{R}, L^p(\mathbb{R}^n))$ norm and $s_\delta \in C_0^\infty(\mathbb{R} \times \mathbb{R}^n)$.

(c) First off, we observe that

$$\|S_\delta u\|^q_{L^q(\mathbb{R}, L^p(\mathbb{R}^n))} = \int_{\mathbb{R}} \|T_\delta u(t)\|^q_{L^p(\mathbb{R}^n)} \, \mathrm{d}t \leq \int_{\mathbb{R}} \|u(t)\|^q_{L^p(\mathbb{R}^n)} \, \mathrm{d}t = \|u\|_{L^q(\mathbb{R}, L^p(\mathbb{R}^n))}$$

where we used Young's inequality to prove that $\|T_\delta\|_{\mathcal{L}(L^p(\mathbb{R}^n))} \leq 1$. We already know that the convergence works on $C_0^\infty(\mathbb{R} \times \mathbb{R}^n)$ which sits dense in $L^q(\mathbb{R}, L^p(\mathbb{R}^n))$.

Furthermore, we have for any Banach space X that on $L^q(\mathbb{R}, X)$, we have

$$\|\alpha_\delta * u\|_{L^q(\mathbb{R},X)} \leq \|\alpha_\delta\|_{L^1(\mathbb{R})} \|u\|_{L^q(\mathbb{R},X)} \leq \|u\|_{L^q(\mathbb{R},X)}$$

which is established in the same way as Young's inequality with $X = \mathbb{R}$ and using $\|\int_{\mathbb{R}} u(t)\ \mathrm{d}t\|_X \leq \int_{\mathbb{R}} \|u(t)\|\ \mathrm{d}t$ by (5.1.9).

In our case, $X = L^p(\mathbb{R}^n)$ and so we get

$$\|\alpha_\delta * (S_\delta u)\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq \|S_\delta u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq \|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))}.$$

So by Banach–Steinhaus, we have immediately that the wanted convergence holds true for every element of $L^q(\mathbb{R}, L^p(\mathbb{R}^n))$.

(d) This is now the same argument as to prove that $C_0^\infty(\mathbb{R}^n)$ lies dense $W^{2,p}(\mathbb{R}^n)$ or $C_0^\infty(\mathbb{R})$ lies dense in $W^{1,q}(\mathbb{R}^n)$ i.e first truncate and then use mollifiers as in (c) which converge together with their derivatives in $L^q(\mathbb{R}, L^p(\mathbb{R}^n))$. So we have indeed that $C_0^\infty(\mathbb{R} \times \mathbb{R}^n)$ is dense in $W^{1,q}(\mathbb{R}, L^p(\mathbb{R}^n)) \cap L^q(\mathbb{R}, W^{2,p}(\mathbb{R}^n))$.

14.2. Hardy's inequality [1] Fix $1 < p < \infty$ and $a > 0$.

(a) Let $f : (0, \infty) \to \mathbb{R}$ be a Lebesgue measurable and suppose that the function $(0, \infty) \to \mathbb{R} : x \to x^{p-1-a} |f(x)|^p$ is integrable. Show that the restriction of f to each interval $(0, x]$ is integrable and prove Hardy's inequality

$$\left(\int_0^\infty x^{-1-a} \left|\int_0^x f(t)\mathrm{d}t\right|^p\ \mathrm{d}x\right)^{\frac{1}{p}} \leq \frac{p}{a}\left(\int_0^\infty x^{p-1-a} |f(x)|^p\ \mathrm{d}x\right)^{\frac{1}{p}}.$$

Show that equality in Hardy's inequality holds if and only if $f = 0$ almost everywhere.

Hint: Assume first that f is nonnegative with compact support and define

$$F(x) := \frac{1}{x}\int_0^x f(t)\ \mathrm{d}t \qquad \text{for } x > 0.$$

Use integration by parts to obtain

$$\int_0^\infty x^{p-1-a} F(x)^p\ \mathrm{d}x = \frac{p}{a}\int_0^\infty x^{p-1-a} F(x)^{p-1} f(x)\ \mathrm{d}x.$$

Use Hölder inequality.

(b) Show that the constant $\frac{p}{a}$ in Hardy's inequality is sharp.

Hint: Choose $\lambda < 1 - \frac{a}{p}$ and take $f(x) = x^{-\lambda}$ for $x \leq 1$ and $f(x) := 0$ for $x > 1$.

(c) Let $f : (0, \infty) \to \mathbb{R}$ be Lebesgue measurable and $|f|^p$ be Lebesgue integrable. Prove that

$$\int_0^\infty \left|\frac{1}{x}\int_0^x f(t)\ \mathrm{d}t\right|^p\ \mathrm{d}x \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty |f(x)|^p\ \mathrm{d}x.$$

[1] This is part of exercise 4.52 p:171 in Dietmar's Measure and Integration book.

(d) Let $f : (0,\infty) \to \mathbb{R}$ be a Lebesgue measurable and suppose that the function $(0,\infty) \to \mathbb{R} : x \to x^{p-1+a} |f(x)|^p$ is integrable. Show that the restriction to each interval $[x,\infty)$ is integrable and prove the inequality

$$\left(\int_0^\infty x^{a-1} \left|\int_x^\infty f(t)\mathrm{d}t\right|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \leq \frac{p}{a}\left(\int_0^\infty x^{p-1+a} |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}}.$$

Hint: Apply the inequality in (a) to the function $g(x) := x^{-2} f(x^{-1})$.

Solution:

(a) Take $\frac{1}{p} + \frac{1}{q} = 1$ i.e. $q = \frac{p}{p-1}$. We start by proving that the restriction of f to each interval $(0,x]$ is integrable. Indeed,

$$\begin{aligned}\int_0^x |f(t)|\,\mathrm{d}t &= \int_0^x t^{\frac{1+a-p}{p}} t^{\frac{p-1-a}{p}} |f(t)|\,\mathrm{d}t \\ &\leq \left(\int_0^x t^{p-1-a} |f(t)|^p \,\mathrm{d}t\right)^{1/p} \left(\int_0^x t^{\frac{a}{p-1}-1}\right)^{1/q} \leq C x^{a/p} < \infty.\end{aligned}$$

Now we follow the hint and assume f positive ($\geq$) , continuous and compactly supported, and calculate

$$\begin{aligned}\int_0^\infty x^{p-1-a} F(x)^p \,\mathrm{d}x &= \int_0^\infty x^{-1-a} (\int_0^x f(t)\,\mathrm{d}t)^p \,\mathrm{d}x \\ &= -\left[\left(\int_0^x f(t)\,\mathrm{d}t\right)^p x^{-a}\frac{1}{a}\right]_0^\infty + \frac{p}{a}\int_0^\infty x^{p-1-a} F(x)^{p-1} f(x)\,\mathrm{d}x.\end{aligned}$$

We want the boundary terms to be zero. For $x = 0$, we have that

$$\left|\left(\int_0^x f(t)\,\mathrm{d}t\right)^p x^{-a}\right| \leq \int_0^x t^{p-1-a} |f(t)|^p \,\mathrm{d}t \to 0$$

as $x \to 0$ due to the integrability assumption. As f is assumed to have compact support, we have that $\operatorname{supp} f \subset (0,L]$ and so f is integrable, by the previous argument. This gives us for $x = \infty$, that for $x > L$,

$$\left|\left(\int_0^x f(t)\,\mathrm{d}t\right)^p x^{-a}\right| \leq \|f\|_{L^1((0,\infty))}^p x^{-a} \to 0$$

as $x \to \infty$. Hence the equality in the hint is proven.

We now can readily derive Hardy's inequality from this inequality, by recognising as in the calculation above that

$$\int_0^\infty x^{p-1-a} F(x)^p \,\mathrm{d}x = \int_0^\infty x^{-1-a}\left(\int_0^x f(t)\,\mathrm{d}t\right)^p \,\mathrm{d}x$$

which is the right hand side of Hardy's inequality to the power p. By the equality, we estimate

$$\begin{aligned}\int_0^\infty x^{p-1-a}F(x)^p \, \mathrm{d}x &= \frac{p}{a}\int_0^\infty x^{p-1-a}F(x)^{p-1}f(x)\,\mathrm{d}x \\ &= \frac{p}{a}\int_0^\infty x^{\frac{p-1-a}{q}}F(x)^{p-1}x^{\frac{p-1-a}{p}}f(x)\,\mathrm{d}x \\ &\le \frac{p}{a}\left(\int_0^\infty x^{p-1-a}f(x)^p\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_0^\infty x^{p-1-a}F(x)^p\,\mathrm{d}x\right)^{\frac{1}{q}}\end{aligned}$$

where we used Hölder inequality, $f > 0$ and $(p-1)q = p$.

The right hand side of Hardy's inequality is finite due to f having compact support, so we may divide in the last inequality, to get

$$\left(\int_0^\infty x^{p-1-a}F(x)^p\,\mathrm{d}x\right)^{\frac{1}{p}} \le \frac{p}{a}\left(\int_0^\infty x^{p-1-a}f(x)^p\,\mathrm{d}x\right)^{\frac{1}{p}}.$$

This establishes the result for f positive, continuous and compactly supported.

Assume that f is only positive. Then approximate f by positive compactly supported functions $f_k \in C_0((0,\infty))$ in $L^p((0,\infty), t^{p-1-a}\,\mathrm{d}t)$ for $k \in \mathbb{N}$, i.e. such that

$$\lim_{k\to\infty}\int_0^\infty t^{p-1-a}\,|f(t) - f_k(t)|^p\,\mathrm{d}t = 0.$$

Now we have that by defining $G(x) := x^{-\frac{1}{p}-\frac{a}{p}}\int_0^x f(t)\,\mathrm{d}t$ and $G_k(x) := x^{-\frac{1}{p}-\frac{a}{p}}\int_0^x f_k(t)\,\mathrm{d}t$.

$$\begin{aligned}&|G(x) - G_k(x)| \le x^{-\frac{1}{p}-\frac{a}{p}}\int_0^x |f(t) - f_k(t)|\,\mathrm{d}t \\ &\le x^{-\frac{1}{p}-\frac{a}{p}}\left(\int_0^x |f(t)-f_k(t)|^p\,t^{p-1-a}\,\mathrm{d}t\right)^{1/p}\left(\int_0^x t^{\frac{a}{p-1}-1}\right)^{1/q} \\ &\le x^{-\frac{1}{p}-\frac{a}{p}}\left(\int_0^x |f(t)-f_k(t)|^p\,t^{p-1-a}\,\mathrm{d}t\right)^{1/p}\frac{p-1}{a}x^{\frac{a}{p}} \\ &\le x^{-\frac{1}{p}}\epsilon\frac{p-1}{a}\end{aligned}$$

for k sufficiently big. So we have pointwise convergence for G. By Fatou's Lemma this means that

$$\begin{aligned}\int_0^\infty G(x)\,\mathrm{d}x &= \int_0^\infty \lim_{k\to\infty} G_k(x)\,\mathrm{d}x \le \lim_{k\to\infty}\int_0^\infty G_k(x)\,\mathrm{d}x \\ &\le \frac{p}{a}\lim_{k\to\infty}\int_0^\infty x^{p-a-1}f_k(x)^p\,\mathrm{d}x = \frac{p}{a}\left(\int_0^\infty x^{p-1-a}f(x)^p\,\mathrm{d}x\right)^{\frac{1}{p}}.\end{aligned}$$

The result is easily extended to general functions f by using the established inequality for the positive function $|f|$.

For the equality case, we already need by the last step that $|\int_0^x f(t)\ \mathrm{d}t| = \int_0^x |f(t)|\ \mathrm{d}t$ which means f is positive or negative. Next, for positive functions, we have that the Hölder step is an equality (The equality of the hint has to hold in the equality cases) only if there is a constant $c > 0$ such that $t^{p-1-a}F(x)^p = ct^{p-1-a}f(x)^p$ almost everywhere, i.e.

$$\frac{1}{x}\int_0^x f(t)\ \mathrm{d}t = c^{\frac{1}{p}} f(x).$$

Hence, f is continuous and differentiable almost everywhere and

$$c^{\frac{1}{p}} f'(x) = -\frac{1}{x^2}\int_0^x f(t)\ \mathrm{d}t + \frac{1}{x}f(x) = (1 - c^{\frac{1}{p}})\frac{1}{x}f(x).$$

Hence, $f(x) = f(1)x^{(1-c^{\frac{1}{p}})/c^{\frac{1}{p}}}$ which has only the prescribed properties for $f(1) = 0$. The same is true for negative functions.

(b) We use for $\lambda < 1 - \frac{a}{p}$ that

$$f(x) = \begin{cases} x^{-\lambda}, \text{ for } & 0 < x < 1 \\ 0, \text{ for } & x \geq 1 \end{cases}$$

Then, we have that

$$\int_0^\infty t^{p-1-a}\,|f(t)|^p\ \mathrm{d}t = \int_0^1 t^{p-1-a-p\lambda}\ \mathrm{d}t = \frac{1}{p-a-p\lambda} < \infty$$

On the other hand, we have

$$\begin{aligned}\int_0^\infty x^{-1-a}\left(\int_0^x f(t)\ \mathrm{d}t\right)^p\ \mathrm{d}x &= \left(\frac{1}{1-\lambda}\right)^p\left(\int_0^1 x^{p-1-a-p\lambda}\ \mathrm{d}x + \int_1^\infty x^{-1-a}\ \mathrm{d}x\right) \\ &= \left(\frac{1}{1-\lambda}\right)^p\left(\frac{1}{p-a-p\lambda} + \frac{1}{a}\right)\end{aligned}$$

Now by comparing, we see that

$$\frac{\int_0^\infty x^{-1-a}\left(\int_0^x f(t)\ \mathrm{d}t\right)^p\ \mathrm{d}x}{\int_0^\infty t^{p-1-a}\,|f(t)|^p\ \mathrm{d}t} = \left(\frac{1}{1-\lambda}\right)^p\left(1 + (p-a-p\lambda)\frac{1}{a}\right)$$

This last expression is an increasing function which converges as $\lambda \to \left(1 - \frac{a}{p}\right)^-$ to $\left(\frac{a}{p}\right)^p$. Hence this is the best constant in Hardy's inequality.

(c) Take $a = p - 1$ in (a).

(d) We may start with $f \geq 0$ as before. We set $g(t) := t^{-2} f(t^{-1})$ and compute

$$\int_0^\infty g(t)^p t^{p-a-1} \, \mathrm{d}t = \int_0^\infty f(t^{-1})^p t^{-p-a-1} \, \mathrm{d}t = \int_0^\infty f(x) x^{p+a+1} x^{-2} \, \mathrm{d}x < \infty$$

Hence, we can use (a), to get

$$\begin{aligned}\int_0^\infty x^{a-1} \left| \int_x^\infty f(t) \, \mathrm{d}t \right|^p \, \mathrm{d}x &= \int_0^\infty x^{-(a-1)-2} \left(\int_0^x g(t) \, \mathrm{d}t \right)^p \, \mathrm{d}x \\ &\leq \left(\frac{p}{a}\right)^p \int_0^\infty t^{p-1-a} g(t)^p \, \mathrm{d}t = \left(\frac{p}{a}\right)^p \int_0^\infty f(x) x^{p+a-1} \, \mathrm{d}x\end{aligned}$$

where in the first equality we used the simultaneous change of variables $t \to t^{-1}$ and $x \to x^{-1}$.

Again as before, the general case can be gotten from the same inequality for its absolute value.

14.3. Uniform maximal regularity of S and its dual.

Prove that for $1 < q, q^* < \infty$ and X a reflexive complex Banach space with $\frac{1}{q} + \frac{1}{q^*} = 1$, the following are equivalent.

- S is uniformly maximal q-regular.
- S^* is uniformly maximal q^*-regular.

where S^* is the dual strongly continuous semigroup (7.3.9).

Hint: Prove by passing to the limit that for $g : [0, T] \to X^*$, $f : [0, T] \to X$ C^1, we have

$$\begin{aligned}&\int_0^T \left\langle g(t), A \int_0^t S(t-s) f(s) \, \mathrm{d}s \right\rangle \, \mathrm{d}t \\ &= \int_0^T \left\langle A^* \int_0^s S^*(s-t) g(T-t) \, \mathrm{d}t, f(T-s) \right\rangle \, \mathrm{d}s.\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the usual pairing between X and its dual.

Solution: Let $h > 0$, $f : [0, T] \to X$ be a continuously differentiable function and

$g : [0, T] \to X^*$ be a continuously differentiable function. Then we calculate

$$\begin{aligned}
&\int_0^T \left\langle g(t), S(h) \int_0^t S(t-s) f(s)\, \mathrm{d}s - \int_0^t S(t-s) f(s)\, \mathrm{d}s \right\rangle \, \mathrm{d}t \\
&= \int_0^T \left\langle g(t), \int_0^t (S(t-s+h) - S(t-s)) f(s)\, \mathrm{d}s \right\rangle \, \mathrm{d}t \\
&= \int_0^T \int_0^t \langle g(t), (S(t-s+h) - S(t-s)) f(s) \rangle \, \mathrm{d}s\, \mathrm{d}t \\
&= \int_0^T \int_s^T \langle (S^*(t-s+h) - S^*(t-s)) g(t), f(s) \rangle \, \mathrm{d}t\, \mathrm{d}s \\
&= \int_0^T \int_0^s \langle (S^*(s-t+h) - S^*(s-t)) g(T-t), f(T-s) \rangle \, \mathrm{d}t\, \mathrm{d}s \\
&= \int_0^T \left\langle S^*(h) \int_0^s S^*(s-t) g(T-t)\, \mathrm{d}t - \int_0^s S^*(s-t) g(T-t)\, \mathrm{d}t, f(T-s) \right\rangle \mathrm{d}s
\end{aligned}$$

where the first two and the last equality follow by (5.1.9), in the third equality we use Fubini, in the fourth equality we use the change of variables $s^* = T - s, t^* = T - t$. Now dividing this equality by h and passing to the limit as $h \to 0$, we get the wanted equality

$$\begin{aligned}
&\int_0^T \left\langle g(t), A \int_0^t S(t-s) f(s)\, \mathrm{d}s \right\rangle \, \mathrm{d}t \\
&= \int_0^T \left\langle A^* \int_0^s S^*(s-t) g(T-t)\, \mathrm{d}t, f(T-s) \right\rangle \, \mathrm{d}s.
\end{aligned}$$

Now assume that first that S^* is is uniformly maximal q^*-regular. Then we can estimate

$$\begin{aligned}
&\left| \int_0^T \left\langle g(t), A \int_0^t S(t-s) f(s)\, \mathrm{d}s \right\rangle \, \mathrm{d}t \right| \\
&= \left| \int_0^T \left\langle A^* \int_0^s S^*(s-t) g(T-t)\, \mathrm{d}t, f(T-s) \right\rangle \, \mathrm{d}s \right| \\
&\leq \int_0^T \left| \left\langle A^* \int_0^s S^*(s-t) g(T-t)\, \mathrm{d}t, f(T-s) \right\rangle \right| \, \mathrm{d}s \\
&\leq \int_0^T \left\| A^* \int_0^s S^*(s-t) g(T-t)\, \mathrm{d}t \right\|_{X^*} \| f(T-s) \|_X \, \mathrm{d}s \\
&\leq \left(\int_0^T \left\| A^* \int_0^s S^*(s-t) g(T-t)\, \mathrm{d}t \right\|_{X^*}^{q^*} \, \mathrm{d}s \right)^{1/(q^*)} \left(\int_0^T \| f(T-s) \|_X^q \, \mathrm{d}s \right)^{1/q} \\
&\leq \|g\|_{L^{q^*}([0,T],X^*)} \|f\|_{L^q([0,T],X)}
\end{aligned}$$

where the third line follows from (5.1.9), the penultimate inequality came from Hölder inequality and the last one from S^* is is uniformly maximal q^*-regular.

As $C^1([0,T],X^*) \subset L^{q^*}([0,T],X^*)$ is dense, we have for all $g \in L^{q^*}([0,T],X^*)$ that

$$\left|\int_0^T \left\langle g(t), A\int_0^t S(t-s)f(s)\ \mathrm{d}s\right\rangle\ \mathrm{d}t\right| \leq \|g\|_{L^{q^*}([0,T],X^*)}\,\|f\|_{L^q([0,T],X)}\,. \tag{1}$$

It can be shown (we leave it as a longer exercise) that $(L^q([0,T],X))^* = L^{q^*}([0,T],X^*)$ and the identification works via the natural pairing

$$L^{q^*}([0,T],X^*) \to (L^q([0,T],X))^* : g \mapsto (f \mapsto \int_0^T \langle g(t), f(t)\rangle\,\mathrm{d}t).$$

In the same way, $(L^{q^*}([0,T],X^*))^* = L^q([0,T],X^{**}) = L^q([0,T],X)$. Hence, by (1) and the definition of the dual norm, we have

$$\left\|A\int_0^t S(t-s)f(s)\ \mathrm{d}s\right\|_{L^q([0,T],X)} \leq \|f\|_{L^q([0,T],X)}\,.$$

This exactly means that S is uniformly maximal q-regular as f was arbitrary. The converse comes from $S^{**} = S$, as we have a reflexive space X.

14.4. Sobolev–Slobodeckij space Let $n \in \mathbb{N}$ and fix real numbers $p \geq 1$ and $0 < s < 1$. The completion of $C_0^\infty(\mathbb{R}^n,\mathbb{C})$ with respect to the norm

$$\|f\|_{w^{s,p}} := \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\ \mathrm{d}y\ \mathrm{d}x\right)^{1/p}$$

is called the homogeneous Sobolev–Slobodeckij space and is denoted by $w^{s,p}(\mathbb{R}^n,\mathbb{C})$. On the other hand, the completion of $C_0^\infty(\mathbb{R}^n,\mathbb{C})$ with respect to the norm

$$\|f\|_{b^{s,p}_{p,1,int}} := \left(\int_0^\infty \frac{1}{r^{sp+1}}\frac{1}{\mu(B_r)}\int_{B_r}\int_{\mathbb{R}^n} |f(x+h)-f(x)|^p\ \mathrm{d}x\ \mathrm{d}h\ \mathrm{d}r\right)^{1/p}$$

is called the homogeneous Besov space and denoted by $b^{s,p}_p(\mathbb{R}^n,\mathbb{C})$.

Prove that $w^{s,p}(\mathbb{R}^n,\mathbb{C}) = b^{s,p}_p(\mathbb{R}^n,\mathbb{C})$.

Hint: Prove that $\|f\|_{b^{s,p}_{p,1,int}} = \left(\frac{1}{(n+sp)\mu(B_1)}\right)^{1/p}\|f\|_{w^{s,p}}$ by using Fubini and spherical coordinates.

Solution: For $x \in \mathbb{R}^n$ and $r > 0$ define $S_r(x) := \{y \in \mathbb{R}^n : |y - x| = r\}$. Then

$$\begin{aligned}
\|f\|^p_{b^{s,p}_{p,1,int}} &= \int_0^\infty \frac{1}{r^{sp+1}} \frac{1}{\mu(B_r)} \int_{B_r} \int_{\mathbb{R}^n} |f(x+h) - f(x)|^p \, \mathrm{d}x \, \mathrm{d}h \, \mathrm{d}r \\
&= \frac{1}{\mu(B_1)} \int_{\mathbb{R}^n} \int_0^\infty \frac{1}{r^{n+sp+1}} \int_{B_r(x)} |f(x) - f(y)|^p \, \mathrm{d}y \, \mathrm{d}r \, \mathrm{d}x \\
&= \frac{1}{\mu(B_1)} \int_{\mathbb{R}^n} \int_0^\infty \int_0^r \frac{1}{r^{n+sp+1}} \int_{S_\rho(x)} |f(x) - f(y)|^p \, \mathrm{d}\sigma(y) \rho^{n-1} \mathrm{d}\rho \mathrm{d}r \mathrm{d}x \\
&= \frac{1}{\mu(B_1)} \int_{\mathbb{R}^n} \int_0^\infty \int_\rho^\infty \frac{1}{r^{n+sp+1}} \int_{S_\rho(x)} |f(x) - f(y)|^p \, \mathrm{d}\sigma(y) \rho^{n-1} \, \mathrm{d}r \mathrm{d}\rho \mathrm{d}x \\
&= \frac{1}{(n+sp)\mu(B_1)} \int_{\mathbb{R}^n} \int_0^\infty \frac{1}{\rho^{n+sp}} \int_{S_\rho(x)} |f(x) - f(y)|^p \, \mathrm{d}\sigma(y) \rho^{n-1} \, \mathrm{d}\rho \, \mathrm{d}x \\
&= \frac{1}{(n+sp)\mu(B_1)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, \mathrm{d}y \, \mathrm{d}x \\
&= \frac{1}{(n+sp)\mu(B_1)} \|f\|^p_{w^{s,p}}
\end{aligned}$$

14.5. Besov for $1 \leq s < 2$. Fix $p, q \geq 1$. Prove that for $f \in C_0^\infty(\mathbb{R}^n)$ non constant and $1 \leq s < 2$, that

$$\|f\|_{b^{s,p}_{q,1}} = \left(\int_0^\infty \left(\frac{\omega_1(r,f)_p}{r^s} \right)^q \frac{dr}{r} \right)^{1/q} = \infty.$$

where $\omega_1(r,f)_p := \sup_{|h| \leq r} \left(\int_{\mathbb{R}^n} |f(x+h) - f(x)|^p \, \mathrm{d}x \right)^{1/p}$.

So one has to replace $\omega_1(r,f)_p$ with $\omega_2(r,p)_p$ in the case $s \geq 1$.

Hint: Use (12.13) in Lemma 12.9.

Solution: We have by (12.13) of Lemma 12.9, that

$$0 \neq c := \frac{\|\nabla f\|_{L^p(\mathbb{R}^n)}}{n} \leq \liminf_{r \to 0} \frac{\omega_1(r,f)_p}{r} =: l.$$

By definition, there is $\delta > 0$ such that $\left| \inf_{r \leq \delta} \frac{\omega_1(r,f)_p}{r} - l \right| \leq \frac{c}{2}$. Meaning that for $0 < r < \delta$, we have

$$\frac{\omega_1(r,f)_p}{r} \geq \frac{c}{2}$$

Or in other words for $0 < r < \delta$,

$$(\frac{\omega_1(r,f)_p}{r})^q \frac{1}{r^{q(s-1)+1}} \geq \left(\frac{c}{2}\right)^q \frac{1}{r^{q(s-1)+1}}$$

Hence the positive integral

$$\int_0^\infty \left(\frac{\omega_1(r,f)_p}{r^s}\right)^q \frac{dr}{r}$$

diverges around 0 as $q(s-1)+1 \geq 1$.