1.1. Young's Inequality. Let $1 \leq r, p, q < \infty$ such that

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Take $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. Define the convolution $f * g$ by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(y) g(x-y) \mathrm{d}y.$$

Prove that $f * g \in L^r(\mathbb{R}^n)$ and that

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Deduce that $(L^1(\mathbb{R}^n), *)$ is a Banach algebra without unit.

Hint: Use the Hölder Inequality for three functions with $\frac{1}{r} + \frac{r-p}{rp} + \frac{r-q}{rq} = 1$ for a point-wise estimate and integrate it.

Solution: We estimate

$$\begin{aligned}
|(f * g)(x)| &\leq \int_{\mathbb{R}^n} |f(y)||g(x-y)| \mathrm{d}y \\
&= \int_{\mathbb{R}^n} (|f(y)|^p |g(x-y)|^q)^{1/r} |f(y)|^{1-p/r} |g(x-y)|^{1-q/r} \mathrm{d}y \\
&\leq \left\| (|f(y)|^p |g(x-y)|^q)^{1/r} \right\|_{L^r} \left\| |f(y)|^{1-p/r} \right\|_{L^{\frac{r-p}{rp}}} \left\| |g(x-y)|^{1-q/r} \right\|_{L^{\frac{r-q}{rq}}} \\
&\leq \left(\int_{\mathbb{R}^n} (|f(y)|^p |g(x-y)|^q) \mathrm{d}y \right)^{1/r} \|f\|_{L^p}^{1-p/r} \|g\|_{L^q}^{1-q/r}
\end{aligned}$$

where the penultimate inequality follows by Hölder for three functions.

Thus integrating this estimate we get

$$\begin{aligned}
\|f * g\|_{L^r}^r &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |g(y)|^p |f(x-y)|^q \mathrm{d}y \ \mathrm{d}x \, \|f\|_{L^p}^{r-p} \|g\|_{L^q}^{r-q} \\
&\leq \|g\|_{L^q}^q \|f\|_{L^p}^p \|f\|_{L^p}^{r-p} \|g\|_{L^q}^{r-q} \\
&\leq (\|f\|_{L^p} \|g\|_{L^q})^r
\end{aligned}$$

where the penultimate inequality follows from Fubini. For L^1, look at the case $p = q = r = 1$.

1.2. Harmonic functions on a two dimensional domain. Let $\Omega \subset \mathbb{C}$ be an open, simply connected subset of $\mathbb{C}$.

(a) Let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Prove that $u := \operatorname{Re} f$ and $v := \operatorname{Im} f$ are harmonic, i.e.

$$\Delta v = \Delta u := \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial x^2} v = 0.$$

(b) Let $u : \Omega \to \mathbb{R}$ be a C^2 harmonic function. Prove that there is a function $v : \Omega \to \mathbb{R}$ such that $f = u + iv : \Omega \to \mathbb{C}$ is holomorphic.

(c) Prove that if $u : \Omega \to \mathbb{R}$ is a C^2 harmonic function, then u is analytic.

(d) **(Mean value property)** Prove that if $u : \Omega \to \mathbb{R}$ is C^2 harmonic, then

$$u(z_0) = \int_0^1 u(z_0 + re^{2\pi i t}) \mathrm{d}t$$

whenever $\bar{B}_r(z_0) \subset \Omega$.

(e) **(Maximum principle)** Prove that if $\Omega' \subset \Omega$ is bounded, then for $u : \Omega \to \mathbb{R}$ C^2 harmonic, we have

$$\max_{\Omega'} u = \max_{\partial\Omega'} u$$

Hint: Use theorems about holomorphic functions e.g. Cauchy's theorem. For (b), consider $G := \partial_x u - i\partial_y u$ and define $v(z) := \operatorname{Im} \int_\gamma G$, where $z_0 \in \Omega$ and $\gamma : [0,1] \to \Omega$ is a smooth path such that $\gamma(0) = z_0$ and $\gamma(1) = z$.

Solution: **(a)** As $f = u + iv$ is holomorphic, the Cauchy-Riemann equations hold.

$$\Delta u = \partial_x \partial_x u + \partial_y \partial_y u = \partial_x \partial_y v - \partial_y \partial_x v = 0.$$

Similarly for v.

(b) Define $G := \partial_x u + -i\partial_y u$. Then we have

$$\partial_y \operatorname{Re} G = \partial_y \partial_x u = \partial_x \partial_y u = -\partial_x \operatorname{Im} G$$
$$\partial_x \operatorname{Re} G = \partial_x \partial_x u = -\partial_y \partial_y u = \partial_y \operatorname{Im} G$$

Hence, G is holomorphic. By Cauchy's theorem and Ω simply connected, we have

$$\int_\gamma G = 0 \tag{1}$$

for every loop $\gamma : S^1 \to \Omega$. Now fix $z_0 \in \Omega$ and define $v : \Omega \to \mathbb{R}$ by choosing for every point $z \in \Omega$ a path $\gamma : [0,1] \to \Omega$ where $\gamma(0) = z_0$ and $\gamma(1) = z$, and

$$v(z) = \operatorname{Im} \int_\gamma G.$$

This function is well-defined by (1) and we calculate for $t \in \mathbb{R} \setminus \{0\}$

$$v(z+t) - v(z) = \operatorname{Im} \int_0^t G(z+\tau)\, \mathrm{d}\tau = \int_0^t (-\partial_y u(z+\tau))\, \mathrm{d}\tau$$
$$v(z+ti) - v(z) = \operatorname{Im} \int_0^t G(z+\tau i) \cdot i\, \mathrm{d}\tau = \int_0^t \partial_x u(z+\tau i)\, \mathrm{d}\tau$$

Hence dividing by t and taking the limit for $t \to 0$, we get

$$\partial_x v = -\partial_y u$$
$$\partial_y v = \partial_x u.$$

Hence $v : \Omega \to \mathbb{R}$ is C^2 harmonic and $f = u + iv$ is holomorphic. u and v are called harmonic conjugates.

(c) By (b), we can find a holomorphic function $f : \Omega \to \mathbb{C}$ such that $u = \operatorname{Re} f$. Hence, u is analytic, as f is analytic.

(d) Let $f : \Omega \to \mathbb{C}$ be holomorphic, such that $u = \operatorname{Re} f$. As f has the mean value property by Cauchy's theorem, so does u. Indeed,

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma_{(z_o,r)}} \frac{f(z)}{(z-z_0)} = \frac{1}{2\pi i} \int_0^1 \frac{f(z_0 + re^{2\pi it})}{re^{2\pi i}} 2\pi i r e^{2\pi it}\, \mathrm{d}t = \int_0^1 f(z_0 + re^{2\pi it})\, \mathrm{d}t$$

Therefore,

$$u(z_0) = \operatorname{Re} f(z_0) = \operatorname{Re} \int_0^1 f(z_0 + re^{2\pi it})\, \mathrm{d}t = \int_0^1 u(z_0 + re^{2\pi it}) \mathrm{d}t$$

(e) u is continuous and $\overline{\Omega'}$ is compact, so u attains a maximum on $\overline{\Omega'}$. Assume the maximum of u on Ω' is attained at any point z_0 of the interior of Ω', then there is a small ball $\bar{B}_r(z) \subset \Omega'$ and on this ball u would violate the mean value property. Therefore the maximum must be attained at a boundary point.

1.3. Symmetries of PDE

(a) Prove that for $O \in \mathrm{O}(n)$ and $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ C^2 harmonic, then

$$v_O(x) := u(Ox)$$

is also harmonic where Ω is open and $x \in \Omega_O := \{x \in \mathbb{R}^n : Ox \in \Omega\}$.

(b) Prove that for $u : \Omega \subset \mathbb{R} \oplus \mathbb{R}^n \to \mathbb{R}$ a C^2 solution of the heat equation i.e.

$$\partial_t u - \Delta_x u = 0$$

where $(t, x) \in \mathbb{R} \oplus \mathbb{R}^n$ and Ω open,

$$v_{\lambda,O}(t, x) = u(\lambda^2 t, \lambda O x)$$

is also a solution of the heat equation for $\lambda > 0$, $O \in \mathrm{O}(n)$ and

$$(t, x) \in \Omega_{\lambda,O} := \{(t, x) \in \mathbb{R} \oplus \mathbb{R}^n : (\lambda^2 t, \lambda O x) \in \Omega\}.$$

(c) [1] Prove that for $u : \Omega \subset \mathbb{R} \oplus \mathbb{R}^n \to \mathbb{R}$ a C^2 solution of the heat equation, then

$$v_\epsilon(t, x) := \frac{1}{\left(\sqrt{1 + 4\epsilon t}\right)^n} \exp\left(\frac{-\epsilon \|x\|^2}{1 + 4\epsilon t}\right) u\left(\frac{t}{1 + 4\epsilon t}, \frac{x}{1 + 4\epsilon t}\right)$$

is also a solution of the heat equation for $\epsilon > 0$ and

$$(t, x) \in \Omega_\epsilon := \{(t, x) \in \mathbb{R} \oplus \mathbb{R}^n : t > -(4\epsilon)^{-1}, (\frac{t}{1 + 4\epsilon t}, \frac{x}{1 + 4\epsilon t}) \in \Omega\}.$$

Use this symmetry starting from the constant solution to get a non-trivial solution v_ϵ of the heat equation. Analyse the behaviour of v_ϵ as $t \to -(4\epsilon)^{-1}$.

Solution:

(a) We can easily compute that the Hessian transforms as follows

$$(\mathrm{d}^2 v_O)(x) = O^\top (\mathrm{d}^2 u)(Ox) O$$

and therefore

$$\Delta v_O(x) = \mathrm{tr}\,(\mathrm{d}^2 v_O(x)) = \mathrm{tr}\,(O^\top \mathrm{d}^2 u(Ox) O) = \mathrm{tr}\,(O O^\top (\mathrm{d}^2 u)(Ox)) = (\Delta u)(Ox) = 0$$

(b) We compute

$$\partial_t v_{\lambda,O}(t, x) = \lambda^2 (\partial_t u)(\lambda^2 t, \lambda O x)$$

and

$$\Delta_x v_{\lambda,O}(t, x) = \lambda^2 (\Delta_x u)(\lambda^2 t, \lambda O x)$$

[1] Thank you Yannick Krifka for pointing out a mistake in a previous version of this exercise.

where we used the calculation from (a). Thus

$$\partial_t v_{\lambda,O}(t,x) - \Delta_x v_{\lambda,O}(t,x) = 0.$$

(c) Denote by $s := \frac{1}{\sqrt{1+4\epsilon t}}$ and calculate

$$\begin{aligned}
&\partial_t v_\epsilon(t,x)\\
&\quad= -2n\epsilon s^{n+2}\exp(-\epsilon\|x\|^2 s^2) f(ts^2, xs^2) + s^{n+4}4\epsilon^2\|x\|^2\exp(-\epsilon\|x\|^2 s^2) f(ts^2, xs^2)\\
&\quad- s^{n+4}4\epsilon\exp(-\epsilon\|x\|^2 s^2)\left\langle x, \partial_x f(ts^2, xs^2)\right\rangle + s^{n+4}\exp(-\epsilon\|x\|^2 s^2)\partial_t f(ts^2, xs^2)\\
&\partial_{x_i} v_\epsilon = s^{n+2}(-2\epsilon x_i)\exp(-\epsilon\|x\|^2 s^2) f(ts^2, xs^2) + s^{n+2}\exp(-\epsilon\|x\|^2 s^2)\partial_{x_i} f(ts^2, xs^2)\\
&\partial_{x_i x_i} v_\epsilon(t,x)\\
&\quad= -2\epsilon s^{n+2}\exp(-\epsilon\|x\|^2 s^2) f(ts^2, xs^2) + s^{n+4}4\epsilon^2 x_i^2\exp(-\epsilon\|x\|^2 s^2) f(ts^2, xs^2)\\
&\quad- s^{n+4}4\epsilon x_i\exp(-\epsilon\|x\|^2 s^2)\partial_{x_i} f(ts^2, xs^2) + s^{n+4}\exp(-\epsilon\|x\|^2 s^2)\partial_{x_i x_i} f(ts^2, xs^2)
\end{aligned}$$

for $i = 1, \dots, n$ and where $\partial_x f$ denotes the gradient of f with respect to x. Now sum over i to see that v_ϵ is solution of the heat equation. Starting from $u(x,t) = c \in \mathbb{R}$, we get

$$v_\epsilon(t,x) = \frac{c}{\sqrt{1+4\epsilon t}}\exp\left(\frac{-\epsilon\|x\|^2}{1+4\epsilon t}\right).$$

This is a constant times the fundamental solution, so as $t \to -(4\epsilon)^{-1}$, we see that v_ϵ goes to zero for $x \neq 0$, but that for $x = 0$ there might remain a 'peak'.

1.4. Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a harmonic function and $f : \mathbb{R} \to \mathbb{R}$ a convex function[2], then $f \circ u$ is subharmonic, i.e.

$$\Delta(f \circ u) \geq 0$$

Solution: We estimate

$$\begin{aligned}
\Delta(f\circ u) &= \sum_{i=1}^n \partial_{x_i}\partial_{x_i}(f\circ u)\\
&= \sum_{i=1}^n \partial_{x_i}\left(f'\circ u\ \partial_{x_i} u\right)\\
&= \sum_{i=1}^n \Big(f''\circ u\ (\partial_{x_i} u)^2 + f'\circ u\ \partial_{x_i}\partial_{x_i} u\Big)\\
&= f''\circ u\ |\nabla u|^2\\
&\geq 0
\end{aligned}$$

[2] This means $f''(t) \geq 0$ for $t \in \mathbb{R}$.