2.1. Laplace equation on $C_0^1(\mathbb{R}^n)$

Let[1] $f \in C_0^1(\mathbb{R}^n)$. Define $u = K * f$. Prove that $u \in C^2(\mathbb{R}^n)$ and that $\Delta u = f$.

Hint: We already know this result for $f \in C_0^2(\mathbb{R}^n)$, so you can try to approximate $f \in C_0^1(\mathbb{R}^n)$ by a sequence of $C_0^2(\mathbb{R}^n)$ functions.

2.2. Uniqueness of solution Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set. Assume for $i = 1, \ldots, n$ that $a_i, c \in C^0(\overline{\Omega})$ with $c(x) < 0$ for all $x \in \Omega$. Prove that there is at most one solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to

$$\begin{cases} \Delta u + \sum_{n=1}^{n} a_i \partial_i u + cu = 0 \\ u|_{\partial\Omega} = f \end{cases} \tag{1}$$

with $f \in C^0(\partial\Omega)$.

Hint: Prove that the problem (1) with $f \equiv 0$ has the unique solution $v \equiv 0$ by showing that $\max v \leq 0$ and $\min v \geq 0$ in this case.

2.3. Subharmonic functions

Let $\Omega \subset \mathbb{R}^n$ be an open subset. Prove that the following statements for $u \in C^2(\Omega)$ are equivalent

(i) $\Delta u \geq 0$

(ii) For all $\xi \in \Omega$ and $r > 0$ such that $\overline{B_r(\xi)} \subset \Omega$, we have

$$u(\xi) \leq \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(\xi)} u \, \mathrm{d}S.$$

(iii) For all $\xi \in \Omega$ and $r > 0$ such that $\overline{B_r(\xi)} \subset \Omega$, we have

$$u(\xi) \leq \frac{n}{\omega_n r^n} \int_{B_r(\xi)} u \, \mathrm{d}x.$$

Hint: $(i) \Rightarrow (ii)$ follows directly from a result seen in the course. For $(ii) \Rightarrow (iii)$ integrate with respect to r and for $(iii) \Rightarrow (i)$ argue by contraposition.

[1] $C_0^1(\mathbb{R}^n)$ is the space of functions with continuous first derivative and compact support.

2.4. Symmetries continued. Let $n > 2$ and $u \in C^2(\mathbb{R}^n)$ be harmonic and define v on $\mathbb{R}^n \setminus \{0\}$ by

$$v(x) = \frac{1}{|x|^{n-2}} u\left(\frac{x}{|x|^2}\right).$$

Prove that v is harmonic. What do you get for u a constant function?

2.5. Maximum principle on unbounded domains. Consider the domain $\Omega = \{x \in \mathbb{R}^n : |x| > 1\}$ and a harmonic function $u \in C^2(\overline{\Omega})$ which has the property $\lim_{r\to\infty} \sup_{\partial B_r(0)\cap\Omega} u = 0$.

Prove that $|u|$ attains its maximum and $\max_\Omega |u| = \max_{\partial\Omega} |u|$.

Please hand in your solutions for this sheet by Monday 07/03/2016.