2.1. Laplace equation on $C_0^1(\mathbb{R}^n)$

Let[1] $f \in C_0^1(\mathbb{R}^n)$. Define $u = K * f$. Prove that $u \in C^2(\mathbb{R}^n)$ and that $\Delta u = f$.

Hint: We already know this result for $f \in C_0^2(\mathbb{R}^n)$, so you can try to approximate $f \in C_0^1(\mathbb{R}^n)$ by a sequence of $C_0^2(\mathbb{R}^n)$ functions.

Solution: Let $f_k \in C_0^2(\mathbb{R}^n)$ be a sequence of functions converging to f in $C_0^1(\mathbb{R}^n)$. (Such a sequence exists as you can see by using convolution with smooth mollifiers covered later in this class. We can furthermore multiply f_k with a smooth cut-off function that is equal to 1 on $\operatorname{supp} f$ and which is supported in the 1 shell $\mathcal{N}_1 \operatorname{supp} f := \{x \in \mathbb{R}^n : \operatorname{dist}(x, \operatorname{supp} f) \leq 1\}$, to get that $\operatorname{supp} f_k \subset \mathcal{N}_1 \operatorname{supp} f \subset B_R(0)$ for some $R > 0$.) That is

$$\|f_k - f\|_{C^0} + \|\nabla f_k - \nabla f\|_{C^0} \to 0.$$

Set $u_k := K * f_k$ and we have [2] $\operatorname{supp} u_k \subset \overline{B_{2R}(0)}$. We have that

$$\begin{aligned}\|u_k - u\|_{C^0} + \|\partial_i u_k - K * \partial_i f\|_{C^0} &= \|K * (f_k - f)\|_{C^0} + \|K * (\partial_i f_k - \partial_i f)\|_{C^0} \\ &\leq \|K\|_{L^1(\overline{B_{2R}(0)})} \left(\|f_k - f\|_{C^0} + \|\nabla f_k - \nabla f\|_{C^0}\right) \overset{k\to\infty}{\longrightarrow} 0\end{aligned}$$

where we used Young's inequality ($p = \infty, q = 1, r = \infty$, which is much easier to prove than the case of Exercise 1.1) and the fact that $K \in L^1_{loc}(\mathbb{R}^n)$ i.e. $K \in L^1(L)$ for every $L \subset \mathbb{R}^n$ compact. Hence $u \in C^1(\mathbb{R}^n)$ and $\partial_i u = K * \partial_i f$. Furthermore, we observe that

$$\begin{aligned}\|\partial_i \partial_j u_k - \partial_j K * \partial_i f\|_{C^0} &= \|\partial_i K * (\partial_j f_k - \partial_j f)\|_{C^0} \\ &\leq \|\partial_i K\|_{L^1(\overline{B_{2R}(0)})} \|\partial_j f_k - \partial_j f\|_{C^0} \overset{k\to\infty}{\longrightarrow} 0.\end{aligned}$$

We used that $\partial_i K \in L^1_{loc}(\mathbb{R}^n)$ and so we get $u \in C^2$ and $\partial_j \partial_i u = \partial_j K * \partial_i u$ which is continuous as $\partial_i f$ is. Hence in particular $\Delta u_k \overset{C^0}{\to} \Delta u$. So we have

$$\begin{aligned}\|\Delta u - f\|_{C^0} &\leq \|\Delta u - \Delta u_k\|_{C^0} + \|\Delta u_k - f_k\|_{C^0} + \|f_k - f\|_{C^0} \\ &\leq \|\Delta u - \Delta u_k\|_{C^0} + 0 + \|f_k - f\|_{C^0} \overset{k\to\infty}{\longrightarrow} 0\end{aligned}$$

where we used the fact that for $f_k \in C_0^2$, $\Delta u_k = f_k$. As the left hand side does not depend on k, we have proven

$$\Delta u = f.$$

[1] $C_0^1(\mathbb{R}^n)$ is the space of functions with continuous first derivative and compact support.
[2] This is a fact about the support of convolutions. Check this!

2.2. Uniqueness of solution Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set. Assume for $i = 1, \dots, n$ that $a_i, c \in C^0(\overline{\Omega})$ with $c(x) < 0$ for all $x \in \Omega$. Prove that there is at most one solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to

$$\begin{cases} \Delta u + \sum_{n=1}^{n} a_i \partial_i u + cu = 0 \\ u|_{\partial\Omega} = f \end{cases} \tag{1}$$

with $f \in C^0(\partial\Omega)$.

Hint: Prove that the problem (1) with $f \equiv 0$ has the unique solution $v \equiv 0$ by showing that $\max v \leq 0$ and $\min v \geq 0$ in this case.

Solution: Assume there were two solutions $u_1, u_2 \in C^2(\Omega) \cap C^0(\overline{\Omega})$, then their difference $v = u_1 - u_2$ is a solution to the problem (1) for $f \equiv 0$. Hence, to prove uniqueness, it is enough to prove that $v \equiv 0$. As $\overline{\Omega}$ is compact, $\max v$ is attained in $\overline{\Omega}$.

Assume by contradiction that $\max v > 0$. Then there is $x_0 \in \Omega$ such that $v(x_0) = \max v > 0$. As x_0 is a maximum, we have that

$$\partial_i \partial_j v(x_0) \leq 0 \qquad \text{and} \qquad \partial_i v(x_0) = 0.$$

Hence $\Delta u \leq 0$, and so

$$\Delta v(x_0) + \sum_{n=1}^{n} a_i(x_0) \partial_i v(x_0) + c(x_0) v(x_0) < 0$$

where we used $c < 0$. This is a contradiction to (1).

Similarly, we prove $\min v \geq 0$. Hence, $v \equiv 0$.

2.3. Subharmonic functions

Let $\Omega \subset \mathbb{R}^n$ be an open subset. Prove that the following statements for $u \in C^2(\Omega)$ are equivalent

(i) $\Delta u \geq 0$

(ii) For all $\xi \in \Omega$ and $r > 0$ such that $\overline{B_r(\xi)} \subset \Omega$, we have

$$u(\xi) \leq \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(\xi)} u \, \mathrm{d}S.$$

(iii) For all $\xi \in \Omega$ and $r > 0$ such that $\overline{B_r(\xi)} \subset \Omega$, we have

$$u(\xi) \leq \frac{n}{\omega_n r^n} \int_{B_r(\xi)} u \, \mathrm{d}x.$$

Hint: $(i) \Rightarrow (ii)$ follows directly from a result seen in the course. For $(ii) \Rightarrow (iii)$ integrate with respect to r and for $(iii) \Rightarrow (i)$ argue by contraposition.

Solution: $\boldsymbol{(i) \Rightarrow (ii)}$: In class you saw the identity

$$u(\xi) = \int_{B_r(\xi)} (\psi(|x-\xi|) - \psi(r))\Delta u(x) \, \mathrm{d}x + \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(\xi)} u \, \mathrm{d}S \tag{2}$$

where

$$\psi(r) = \begin{cases} \dfrac{r^{2-n}}{(2-n)\omega_n} & \text{for} \quad n > 2 \\ \dfrac{\log r}{2\pi} & \text{for} \quad n = 2 \end{cases}$$

is the fundamental solution with $r > 0$.

As ψ is decreasing and $\Delta u \geq 0$, we have that the first integral in (2) is ≥ 0, thereby getting the inequality in (ii).

$\boldsymbol{(ii) \Rightarrow (iii)}$: We integrate the inequality

$$\omega_n \rho^{n-1} u(\xi) \leq \int_{\partial B_\rho(\xi)} u \, \mathrm{d}S$$

with respect to ρ from 0 to r to get (iii).

$\boldsymbol{(iii) \Rightarrow (i)}$: Assume there is $\xi_0 \in \Omega$ such that $\Delta u(\xi_0) < 0$. Then as Δu is continuous, there is $r > 0$ such that $\overline{B_r(\xi_0)} \subset \Omega$ and $\Delta u(\xi) < 0$ for all $\xi \in B_r(\xi_0)$. Hence the first integral in (2) is < 0, thus for $0 < s < r$ we have

$$s^{n-1} u(\xi_0) > \frac{1}{\omega_n} \int_{\partial B_s(\xi_0)} u \, \mathrm{d}S$$

and thus integrating this inequality for s from 0 to r, we get

$$u(\xi_0) > \frac{n}{\omega_n r^n} \int_{B_r(\xi_0)} u \, \mathrm{d}x.$$

Hence if (i) does not hold, (iii) also does not hold.

2.4. Symmetries continued. Let $n > 2$ and $u \in C^2(\mathbb{R}^n)$ be harmonic and define v on $\mathbb{R}^n \setminus \{0\}$ by

$$v(x) = \frac{1}{|x|^{n-2}} u\left(\frac{x}{|x|^2}\right).$$

Prove that v is harmonic. What do you get for u a constant function?

Solution: We first calculate

$$\partial_i v(x) = \partial_i \left(\frac{1}{|x|^{n-2}}\right) u + \frac{1}{|x|^{n-2}} (\partial_j u) \partial_i \left(\frac{x_j}{|x|^2}\right)$$

where the argument of u and $\partial_j u$ is always $\frac{x}{|x|^2}$ and we summed over $j \in \{1, \dots, n\}$. Continuing, we get

$$\begin{aligned}\partial_i \partial_i v(x) =& \partial_i \partial_i \left(\frac{1}{|x|^{n-2}}\right) u + 2\partial_i \left(\frac{1}{|x|^{n-2}}\right) (\partial_j u) \partial_j \left(\frac{x_j}{|x|^2}\right) \\ &+ \frac{1}{|x|^{n-2}} (\partial_k \partial_j u) \partial_i \left(\frac{x_k}{|x|^2}\right) \partial_i \left(\frac{x_j}{|x|^2}\right) + \frac{1}{|x|^{n-2}} (\partial_j u) \partial_i \partial_i \left(\frac{x_j}{|x|^2}\right)\end{aligned}$$

where again the arguments are all $\frac{x}{|x|^2}$ and we sum over j and k. Summing over i, we get

$$\begin{aligned}\Delta v(x) =& \Delta \left(\frac{1}{|x|^{n-2}}\right) u + 2 \sum_j \left\langle \nabla \left(\frac{1}{|x|^{n-2}}\right), \nabla \left(\frac{x_j}{|x^2|}\right) \right\rangle (\partial_j u) \\ &+ \frac{1}{|x|^{n-2}} \sum_{j,k} \left\langle \nabla \left(\frac{x_k}{|x^2|}\right), \nabla \left(\frac{x_j}{|x^2|}\right) \right\rangle (\partial_k \partial_j u) + \frac{1}{|x|^{n-2}} (\partial_j u) \Delta \left(\frac{x_j}{|x|^2}\right)\end{aligned}$$

where again the arguments are all $\frac{x}{|x|^2}$. Now that we have put up calculating derivatives, let us do it now

$$\partial_i \left(\sqrt{\sum_i x_i^2}\right) = \frac{1}{2|x|} 2x_i = \frac{x_i}{|x|}.$$

Hence,

$$\begin{aligned}\nabla \left(\frac{1}{|x|^{n-2}}\right) &= -\frac{(n-2)x}{|x|^n} \\ \Delta \left(\frac{1}{|x|^{n-2}}\right) &= 0 \\ \nabla \left(\frac{x_j}{|x^2|}\right) &= \frac{1}{|x|^2} (e_j - 2\frac{x_j}{|x^2|} x) \\ \Delta \left(\frac{x_j}{|x^2|}\right) &= -2(n-2) \frac{x_j}{|x|^4}.\end{aligned}$$

Therefore, we get

$$\Delta v(x) = 0 + 2\sum_j \frac{n-2}{|x|^{n-2}} x_j(\partial_j u) + \frac{1}{|x|^{n-2}}\Delta u - 2\sum_j \frac{n-2}{|x|^{n-2}} x_j(\partial_j u) = \frac{1}{|x|^{n-2}}\Delta u.$$

We conclude that v is harmonic if and only if u is harmonic.

For $u \equiv \text{const}$, we get that v is a multiple of the fundamental solution.

2.5. Maximum principle on unbounded domains. Consider the domain $\Omega = \{x \in \mathbb{R}^n : |x| > 1\}$ and a harmonic function $u \in C^2(\overline{\Omega})$ which has the property $\lim_{r\to\infty} \sup_{\partial B_r(0)\cap\Omega} u = 0$.

Prove that $|u|$ attains its maximum and $\max_\Omega |u| = \max_{\partial\Omega} |u|$.

Solution: We apply the maximum principle for harmonic functions on $B_r(0)$ for $r > 1$, to get

$$\max_{\Omega\cap B_r(0)} |u| = \max(\max_{\partial\Omega} |u|, \max_{\partial B_r(0)} |u|)$$

Let $s = \max_{\partial\Omega} |u|$. If $s > 0$, then we can find $R > 0$ such that $\max_{\partial B_r(0)} |u| < s$ for all $r > R$, and therefore

$$\sup_{B_r(0)} |u| = \lim_{r\to\infty} \max_{\Omega\cap B_r(0)} |u| = \max_{\partial\Omega} |u|.$$

Hence the maximum is attained on $\partial\Omega$ and we can replace sup by max.

If $s = 0$, then we have

$$\sup_{B_r(0)} |u| = \lim_{r\to\infty} \max_{\Omega\cap B_r(0)} |u| = \lim_{r\to\infty} \max_{\partial B_r(0)} |u| = 0.$$

Therefore $|u| \equiv 0$ and the maximum is attained at any point of Ω.