3.1. Green function for the ball in dimension 2 Show that the Green function for the 2-dimensional ball $B_r(0) \subset \mathbb{R}^2$ is given by

$$G_\xi(x) = \frac{1}{2\pi}\left(\log|\xi - x| - \log\left|\frac{r\xi}{|\xi|} - \frac{|\xi|x}{r}\right|\right)$$

for $\xi \in B_r(0)$. From this, prove that the formula for the Poisson kernel $P(\xi, x)$ coincides with the formula for $n > 2$.

Hint: Follow the proof given for the n-dimensional case, $n > 2$.

3.2. Poisson formula in two-dimensional polar coordinates Assume $f : \mathbb{R} \to \mathbb{R}$ is a C^2 periodic function of period 2π with Fourier series

$$f(\theta) = \sum_{n=0}^{\infty}(a_n\cos(n\theta) + b_n\sin(n\theta)).$$

Prove that $u : \overline{B_r(0)} \subset \mathbb{R}^2 \to \mathbb{R}$ given in polar coordinates by

$$u(r,\theta) = \sum_{n=0}^{\infty}\left[(a_n\cos(n\theta) + b_n\sin(n\theta))r^n\right]$$

is harmonic on $B_1(0)$, is in $C^0(\overline{B_1(0)})$ and that $u(1,\theta) = f(\theta)$.

3.3. Green function for the half space. Fix $n \geq 2$. Consider the domain

$$\Omega := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}.$$

(a) Derive the following Poisson formula for $\xi \in \Omega$

$$u(\xi) = \frac{2\xi_n}{\omega_n}\int_{\partial\Omega}\frac{f(y)}{|\xi - y|^n}\,\mathrm{d}y \tag{1}$$

where $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution of

$$\begin{cases} \Delta u(x) = 0 & \text{for } x \in \Omega \\ u|_{\partial\Omega} = f & \text{on } \partial\Omega \end{cases} \tag{2}$$

and $f \in C^0(\partial\Omega)$.

(b) Show that for $f \in L^\infty(\partial\Omega) \cap C^0(\partial\Omega)$, (1) defines a solution in $C^2(\Omega) \cap L^\infty(\Omega)$ of problem (2).

Hint: For (a), use $\xi^* = (\xi_1, \dots, \xi_{n-1}, -\xi_n)$ as reflected point to compute the Green function. For (b), establish $\int_{\partial\Omega} P_\xi(y)\,\mathrm{d}y = 1$, by using (1).

3.4. Harnack's inequality For $\Omega = B_r(0) \subset \mathbb{R}^n$, let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be ≥ 0 and harmonic on Ω. Prove that for $\xi \in \Omega$, one has

$$\frac{r^{n-2}(r - |\xi|)}{(r + |\xi|)^{n-1}} u(0) \leq u(\xi) \leq \frac{r^{n-2}(r + |\xi|)}{(r - |\xi|)^{n-1}} u(0).$$

Hint: Use Poisson's formula.

3.5. Barrier postulate for some special domains Let $\Omega \subset \mathbb{R}^n$ be an open set having the property: for every $\zeta \in \partial\Omega$, there is a ball $B_r(\xi) \in \mathbb{R}^n$ such that

$$\overline{\Omega} \cap \overline{B_r(\xi)} = \{\zeta\}.$$

Show that Ω fulfils the barrier postulate i.e. for every $\zeta \in \partial\Omega$ there exists a subharmonic function $Q_\zeta \in C^0(\overline{\Omega})$ such that

$$Q_\zeta(\zeta) = 0, \qquad Q_\zeta(x) < 0 \qquad \text{for } x \in \partial\Omega, x \neq \zeta.$$

Hint: Use the fundamental solution of the Laplace equation.

Please hand in your solutions for this sheet by Monday 14/03/2016.