3.1. Green function for the ball in dimension 2 Show that the Green function for the 2-dimensional ball $B_r(0) \subset \mathbb{R}^2$ is given by

$$G_\xi(x) = \frac{1}{2\pi}\left(\log|\xi - x| - \log\left|\frac{r\xi}{|\xi|} - \frac{|\xi|x}{r}\right|\right)$$

for $\xi \in B_r(0)$. From this, prove that the formula for the Poisson kernel $P(\xi, x)$ coincides with the formula for $n > 2$.

Hint: Follow the proof given for the n-dimensional case, $n > 2$.

Solution: We can employ the same strategy as in the case $n \geq 3$. Namely, look at $\xi^* = \frac{r^2\xi}{|\xi|^2}$, and verify that for $x \in \partial B_r(0)$, we have

$$\frac{|\xi|}{r}|x - \xi^*| = |x - \xi|\,.$$

Notice, that $\xi^* \notin B_r(0)$, hence $|x - \xi^*| \neq 0$ for all $x \in B_r(0)$. Hence, we can write down the Green function for $B_r(0)$ as

$$G_\xi(x) := \psi(|x-\xi|) - \psi\left(\frac{|\xi|}{r}|x - \xi^*|\right) = \frac{1}{2\pi}\left(\log|x - \xi| - \log\left|\frac{r\xi}{|\xi|} - \frac{|\xi|x}{r}\right|\right).$$

This is indeed the Green function, as it vanishes on $\partial B_r(0)$ and as G_ξ is the sum of the fundamental solution with a smooth harmonic function.

To derive the Poisson kernel, we have to derive G_ξ in the normal direction along the boundary $\partial B_r(0)$. We have for $x \in \partial B_r(0)$, that

$$\begin{aligned}
\partial_{x_i}\psi(|x-\xi|) &= \frac{(x_i - \xi_i)}{2\pi|x-\xi|^2} \\
\partial_{x_i}\psi\left(\frac{|\xi|}{r}|x-\xi^*|\right) &= \frac{(x_i - \xi_i^*)}{2\pi|x-\xi^*|^2} = \frac{(\frac{|\xi|^2}{r^2}x_i - \xi_i)}{2\pi|x-\xi|^2}
\end{aligned}$$

Therefore, we get for $x \in \partial B_r(0)$,

$$\begin{aligned}
P_\xi(x) = \frac{\partial}{\partial n_x}G_\xi(x) = \sum_{i=1}^{2}\frac{x_i}{r}\partial_{x_i}G_\xi(x) &= \sum_{i=1}^{2}\left(\frac{x_i(x_i-\xi_i)}{2\pi r|x-\xi|^2} - \frac{x_i(\frac{|\xi|^2}{r^2}x_i - \xi_i)}{2\pi r|x-\xi|^2}\right) \\
&= \frac{r^2 - |\xi|^2}{2\pi r(|x-\xi|^2)}
\end{aligned}$$

which agrees with the formula for $n \geq 3$.

3.2. Poisson formula in two-dimensional polar coordinates Assume $f : \mathbb{R} \to \mathbb{R}$ is a C^2 periodic function of period 2π with Fourier series

$$f(\theta) = \sum_{n=0}^{\infty} (a_n \cos(n\theta) + b_n \sin(n\theta)).$$

Prove that $u : \overline{B_r(0)} \subset \mathbb{R}^2 \to \mathbb{R}$ given in polar coordinates by

$$u(r,\theta) = \sum_{n=0}^{\infty} [(a_n \cos(n\theta) + b_n \sin(n\theta))r^n]$$

is harmonic on $B_1(0)$, is in $C^0(\overline{B_1(0)})$ and that $u(1,\theta) = f(\theta)$.

Solution: The Laplace equation in polar coordinates is given by

$$\Delta = \partial_r^2 + \frac{\partial_r u}{r} + \frac{\partial_\theta^2 u}{r^2}.$$

Hence, we can calculate

$$\begin{aligned}\Delta(\cos(n\theta)r^n) &= n(n-1)\cos(n\theta)r^{n-2} + \frac{n\cos(n\theta)r^{n-1}}{r} - \frac{n^2\cos(n\theta)r^n}{r^2} \\ &= [n(n-1) + n - n^2]\cos(n\theta)r^{n-2} = 0\end{aligned}$$

and also

$$\Delta(\sin(n\theta)r^n) = 0.$$

Therefore, the partial sums $u_N(r,\theta) = \sum_{n=1}^{N} [(a_n \cos(n\theta) + b_n \sin(n\theta))r^n]$ is smooth and harmonic for every $N \in \mathbb{N}$. As we have that

$$|a_n| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\ \mathrm{d}\theta\right| \leq \sup_{\mathbb{R}} |f|$$

and likewise $|b_n| \leq \sup_{\mathbb{R}} |f|$, we get uniform convergence of u_N on every compact subset of $B_1(0)$. As the uniform limit of harmonic functions is again harmonic, we conclude that u is harmonic on $B_1(0)$.

As $f \in C^2$, we can get better estimates for the coefficients $|a_n|$ and $|b_n|$. Namely,

$$|n^2 a_n| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(\theta) n^2 \cos(n\theta)\ \mathrm{d}\theta\right| = \left|\frac{1}{2\pi}\int_0^{2\pi} f''(\theta)\cos(n\theta)\ \mathrm{d}\theta\right| \leq \sup_{\mathbb{R}} |f''|$$

where we used integration by parts twice in the second equality. Similarly, we have $n^2|b_n| \leq sup_{\mathbb{R}} |f''|$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, u also converges uniformly on $\overline{B_1(0)}$ and in particular to f on the boundary.

3.3. Green function for the half space. Fix $n \geq 2$. Consider the domain

$$\Omega := \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}.$$

(a) Derive the following Poisson formula for $\xi \in \Omega$

$$u(\xi) = \frac{2\xi_n}{\omega_n} \int_{\partial\Omega} \frac{f(y)}{|\xi - y|^n} \, \mathrm{d}y \tag{1}$$

where $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution of

$$\begin{cases} \Delta u(x) = 0 & \text{for } x \in \Omega \\ u|_{\partial\Omega} = f & \text{on } \partial\Omega \end{cases} \tag{2}$$

and $f \in C^0(\partial\Omega)$.

(b) Show that for $f \in L^\infty(\partial\Omega) \cap C^0(\partial\Omega)$, (1) defines a solution in $C^2(\Omega) \cap L^\infty(\Omega)$ of problem (2).

Hint: For (a), use $\xi^* = (\xi_1, \ldots, \xi_{n-1}, -\xi_n)$ as reflected point to compute the Green function. For (b), establish $\int_{\partial\Omega} P_\xi(y) \, \mathrm{d}y = 1$, by using (1).

Solution: **For (a):** We consider the point $\xi^* = (\xi_1, \ldots, \xi_{n-1}, -\xi_n)$ and note that for $x \in \partial\Omega$,

$$|x - \xi| = |x - \xi^*|$$

and therefore we put

$$G_\xi(x) = \psi(|x - \xi|) - \psi(|x - \xi^*|).$$

The function G_ξ is indeed the Green function for Ω as G_ξ vanishes on $\partial\Omega$ and it is sum of the fundamental solution with a smooth harmonic function in Ω. We have [1]

$$\partial_{x_n}\psi(|x - \xi|) = \frac{(x_n - \xi_n)}{\omega_n |x - \xi|^n} \qquad \partial_{x_n}\psi(|x - \xi^*|) = \frac{(x_n - \xi_n^*)}{\omega_n |x - \xi|^n}$$

and so we get

$$P_\xi(x) = \frac{\partial}{\partial n_x} G_\xi(x) = -\partial_{x_n} G_\xi(x) = \frac{2\xi_n}{\omega_n |x - \xi|^n}.$$

Thus we get the formula for a harmonic function $u : \Omega \to \mathbb{R}$ having boundary values given by f,

$$u(\xi) = -\frac{2\xi_n}{\omega_n} \int_{\partial\Omega} \frac{f(y)}{|\xi - y|^n} \, \mathrm{d}y.$$

[1] Note that we didn't have to distinguish $n = 2$ and $n \geq 3$, as the expressions for the derivatives agree.

For (b): First we establish boundedness for u defined by (1). We have the estimate

$$|u(\xi)| \leq \sup_{\partial\Omega} |f| \int_{\partial\Omega} P_\xi(y)\ \mathrm{d}y = \sup_{\partial\Omega} |f|.$$

where the last equality follows from plugging $u = f = 1$ into (1) which is permitted as $u = 1$ solves (2) for $f = 1$. Next, we have that $P_\xi(x)$ is harmonic in ξ on Ω for $x \in \partial\Omega$. Hence, u is smooth and harmonic by differentiation under the sign of integration. Finally, what is left to prove, is that u is continuous up to the boundary and that $u|_{\partial\Omega} = f$.

Therefore, fix $y_0 \in \partial\Omega$ and for $\epsilon > 0$, there is $\delta > 0$ such that $|f(y) - f(y_0)| < \epsilon$ for all $y \in \partial\Omega$ with $|y - y_0| < \delta$. Then for $\xi \in \Omega$ with $|\xi - y_0| < \delta/2$, we get

$$\begin{aligned}
|u(\xi) - f(y_0)| &= \left| -\frac{2\xi_n}{\omega_n} \int_{\partial\Omega} \frac{f(y) - f(y_0)}{|\xi - y|^n}\ \mathrm{d}y \right| \\
&\leq \frac{2\xi_n}{\omega_n} \int_{\partial\Omega} \frac{|f(y) - f(y_0)|}{|\xi - y|^n}\ \mathrm{d}y \\
&= \frac{2\xi_n}{\omega_n} \int_{B_\delta(y_0)\cap\partial\Omega} \frac{|f(y) - f(y_0)|}{|\xi - y|^n}\ \mathrm{d}y + \frac{2\xi_n}{\omega_n} \int_{\partial\Omega\setminus B_\delta(y_0)} \frac{|f(y) - f(y_0)|}{|\xi - y|^n}\ \mathrm{d}y \\
&= I_1 + I_2
\end{aligned}$$

Now for I_1, we have

$$I_1 \leq \epsilon \int_{\partial\Omega} P_\xi(y)\ \mathrm{d}y \leq \epsilon.$$

On the other hand, for I_2, we get for $y \in \partial\Omega \setminus B_\delta(y_0)$

$$|y - y_0| \leq |y - \xi| + |\xi - y_0| \leq |y - \xi| + \frac{\delta}{2} \leq |y - \xi| + \frac{1}{2}|y - y_0|$$

i.e. $\frac{1}{2}|y - y_0| \leq |y - \xi|$. Therefore, we have

$$\begin{aligned}
I_2 &\leq \frac{4\xi_n \|f\|_{L^\infty}}{\omega_n} \int_{\partial\Omega\setminus B_\delta(y_0)} \frac{2^n}{|y_0 - y|^n}\ \mathrm{d}y \\
&\leq \frac{2^{n+2}\xi_n \|f\|_{L^\infty}}{\omega_n} \int_{\partial\Omega\setminus B_\delta(0)} \frac{1}{|y|^n}\ \mathrm{d}y \\
&\leq \frac{2^{n+2}\xi_n \|f\|_{L^\infty}}{\omega_n} \int_\delta^\infty r^{n-2} r^{-n} \int_{S^{n-2}} \mathrm{d}\sigma\ \mathrm{d}r \\
&\leq \frac{2^{n+2}\xi_n \|f\|_{L^\infty}\, \omega_{n-2}}{\omega_n \delta}
\end{aligned}$$

Thus if we further require $|y_0 - \xi| \le \frac{\epsilon\omega_n\delta}{2^{n+2}\|f\|_{L^\infty}\omega_{n-2}}$, we get $I_2 \le \epsilon$. Hence

$$|u(\xi) - f(y_0)| \le 2\epsilon$$

whenever $|\xi - y_0| \le \min(\frac{\delta}{2}, \frac{\epsilon\omega_n\delta}{2^{n+2}\|f\|_{L^\infty}\omega_{n-2}})$, which proves continuity of u at y_0 and extending u by continuity, we have $u(y_0) = f(y_0)$ as desired.

3.4. Harnack's inequality For $\Omega = B_r(0) \subset \mathbb{R}^n$, let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be ≥ 0 and harmonic on Ω. Prove that for $\xi \in \Omega$, one has

$$\frac{r^{n-2}(r-|\xi|)}{(r+|\xi|)^{n-1}}u(0) \le u(\xi) \le \frac{r^{n-2}(r+|\xi|)}{(r-|\xi|)^{n-1}}u(0).$$

Hint: Use Poisson's formula.

Solution: We have that for $y \in \partial B_r(0)$, that for $\xi \in B_r(0)$

$$r - |\xi| \le |\xi - y| \le r + |\xi|.$$

Hence, we have the following bounds for the Poisson kernel

$$\frac{r^{(n-2)}(r-|\xi|)}{(r+|\xi|)^{n-1}}\frac{1}{\omega_n r^{(n-1)}} \le \frac{r^2-|\xi|^2}{r\omega_n|\xi-y|^n} \le \frac{r^{(n-2)}(r+|\xi|)}{(r-|\xi|)^{n-1}}\frac{1}{\omega_n r^{(n-1)}}$$

Thus multiplying with $u(y) \ge 0$ and integrating over $\partial B_r(0)$, we get

$$\begin{aligned}\frac{r^{(n-2)}(r-|\xi|)}{(r+|\xi|)^{n-1}}\frac{1}{\omega_n r^{(n-1)}}\int_{\partial B_r(0)} u(y)\,\mathrm{d}S_y &\le \int_{\partial B_r(0)} \frac{r^2-|\xi|^2}{r\omega_n|\xi-y|^n}u(y)\,\mathrm{d}S_y \\ &\le \frac{r^{(n-2)}(r+|\xi|)}{(r-|\xi|)^{n-1}}\frac{1}{\omega_n r^{(n-1)}}\int_{\partial B_r(0)} u(y)\,\mathrm{d}S_y.\end{aligned}$$

Hence, by Poisson's formula and by mean value theorem, we finally get

$$\frac{r^{n-2}(r-|\xi|)}{(r+|\xi|)^{n-1}}u(0) \le u(\xi) \le \frac{r^{n-2}(r+|\xi|)}{(r-|\xi|)^{n-1}}u(0).$$

3.5. Barrier postulate for some special domains Let $\Omega \subset \mathbb{R}^n$ be an open set having the property: for every $\zeta \in \partial\Omega$, there is a ball $B_r(\xi) \in \mathbb{R}^n$ such that

$$\overline{\Omega} \cap \overline{B_r(\xi)} = \{\zeta\}.$$

Show that Ω fulfils the barrier postulate i.e. for every $\zeta \in \partial\Omega$ there exists a subharmonic function $Q_\zeta \in C^0(\overline{\Omega})$ such that

$$Q_\zeta(\zeta) = 0, \qquad Q_\zeta(x) < 0 \qquad \text{for } x \in \partial\Omega, x \ne \zeta.$$

Hint: Use the fundamental solution of the Laplace equation.

Solution: If $B_r(\xi)$ has the required property for $\zeta \in \partial\Omega$, then consider

$$Q_\zeta(x) := K_\xi(x) - K_\xi(\zeta)$$

where $K_\xi(x) := \psi(|\xi - x|)$ is the fundamental solution for the Laplace equation. We have $Q_\zeta(\zeta) = 0$ and $Q_\zeta(x) < 0$ for all other points x on the boundary as K_ξ is a decreasing function. Also $\Delta_x K_\xi(x) = 0$, therefore Q_ζ is harmonic on Ω which implies subharmonicity.