4.1. Removal of singularities [1] Assume $u : B_1(\xi) \setminus \{\xi\} \subset \mathbb{R}^n \to \mathbb{R}$ be a harmonic function which fulfils the condition

$$\lim_{r \to 0} r^{n-1} \sup_{|\xi - x| = r} \left(|u(x)| + |\nabla u(x)| \right) = 0.$$

Then u can be extended to a harmonic function on $B_1(\xi)$.

Hint: Prove that u is a weak solution for the Laplace equation on $B_1(\xi)$ by cutting out a small ball $B_r(\xi)$. Use Weyl's Lemma.

4.2. Reflection principle [2] Denote by $B_1(0)^+ := \{x \in B_1(0) \subset \mathbb{R}^n : x_n > 0\}$. Assume that $u : B_1(0)^+ \to \mathbb{R}$ is harmonic and admits a continuous extension to $\overline{B_1(0)^+}$ with $u \equiv 0$ on $x_n = 0$. Define an extension $\tilde{u}$ of u to $B_1(0)$ by defining

$$u(x) = -u(\tilde{x}, -x_n)$$

for $x_n < 0$ and where we write $x = (\tilde{x}, x_n) \in \mathbb{R}^{n-1} \oplus \mathbb{R}$. Prove that $\tilde{u} : B_1(0) \to \mathbb{R}$ is harmonic.

Hint: Prove that $\tilde{u}$ is a weak solution for the Laplace equation on $B_1(0)$ by splitting φ into even and uneven parts with respect to x_n and cutting out a symmetric strip around $x_n = 0$. Use Lemma 2 to obtain bounds on ∇u and Weyl's Lemma.

4.3. Injectivity of functions Prove that the inclusion $L^1_{loc}(\Omega) \hookrightarrow \mathcal{D}(\Omega)$ is injective. In other words, prove that any $u \in L^1_{loc}(\Omega)$ with the property $\int_\Omega \varphi u \, \mathrm{d}x = 0$ for all $\varphi \in C_0^\infty(\Omega)$ must be zero almost everywhere.

Hint: This is an exercise in measure theory, and therefore we give extensive hints and references to the script on measure theory by Prof. Salamon. [3] Step 1: Convince yourself, that there is a representative of u which is Borel measurable. Step 2: Prove that $\int_K u = 0$ for all compact sets $K \subset \Omega$ by using cut-off functions and Lebesgue dominated convergence. Step 3: Define two measures $\mu^+(A) = \int_A u^+ \, \mathrm{d}x$ and $\mu^-(A) = \int_A u^- \, \mathrm{d}x$ where $u^+ = \max(u, 0)$ and $u^- = -\min(f, 0)$. Check that these are Borel measures by using Theorem 1.40. Step 4: Use 3.18, to get μ^+ and μ^- are inner regular. Step 5: Prove that μ^+ and μ^- are zero, by decomposing Borel sets A into A^+ and A^- where $u \geq 0$ and $u \leq 0$ and using Step 4 together with Step 2. Step 6: Use Lemma 1.49 to conclude.

[1] Thank you Christian Beck for giving me the idea for this exercise.
[2] This exercise is a bit longer, but has the same general idea as 4.1.
[3] This can be found at `https://people.math.ethz.ch/~salamon/PREPRINTS/measure.pdf`.

4.4. Equivalent norms Let $u \in W^{k,p}(\mathbb{R}^n)$, which means that u has weak derivatives $\partial^\alpha u \in L^p(\mathbb{R}^n)$ for every multi-index $\alpha \in (\mathbb{N} \cup \{0\})^n$ with $|\alpha| \leq k$. Prove that the norms

$$\sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^p(\mathbb{R}^n)} \qquad \text{and} \qquad \left(\sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} |\partial^\alpha u|^p \, \mathrm{d}x \right)^{\frac{1}{p}}$$

are equivalent.

4.5. Reflexivity of Sobolev spaces Prove that $W^{k,p}(\mathbb{R}^n)$ is reflexive for all $k \in \mathbb{N} \cup \{0\}$ and $1 < p < \infty$.

Hint: Recall from FA I, that the spaces $L^p(\mathbb{R}^n, \mathbb{R}^N)$ are reflexive for $1 < p < \infty$.

Please hand in your solutions for this sheet by Monday 21/03/2016.