5.1. Prove that the $L^1_{loc}(\mathbb{R})$ function $u : \mathbb{R} \to \mathbb{R} : x \mapsto |x|$, has a weak derivative in $L^1_{loc}(\mathbb{R})$.

5.2. Weak derivative of K. Let $K := K_0$ be the fundamental solution of the Laplace operator, $n \geq 2$. Prove that the first strong derivative $\partial_i K$ of K, defined everywhere but the origin, is also the first weak derivative of K for $1 \leq i \leq n$.

N.B. Note that this is not true for the second derivatives, as K is not a weak solution for the Laplace equation, but still $\Delta K = 0$ everywhere but the origin.

5.3. Let $I = (a, b) \subset \mathbb{R}$ be a possibly unbounded open interval and let $1 \leq p \leq \infty$. Show that $u \in W^{1,p}(I)$ if and only if u is continuous, $u \in L^p(I)$[1] and there is $v \in L^p(I)$ such that

$$u(t) - u(s) = \int_s^t v(r) \, \mathrm{d}r$$

for all $t, s \in I$.

5.4. Embedding theorem for $n = 1$. Let $I = (a, b)$ be a bounded, open interval in $\mathbb{R}$ and $1 \leq p \leq \infty$. Prove that u is in the Hölder space $C^{0,1-\frac{1}{p}}(I)$ and that for $v \in L^p(I)$ the weak derivative of u, we have

$$\sup_{s,t \in I, t \neq s} \frac{|u(t) - u(s)|}{|t - s|^{1-\frac{1}{p}}} \leq \|v\|_{L^p(I)}$$

Deduce that the immersion $W^{1,p}(I) \to C^0(I)$ is compact for $p > 1$ and find a counterexample to compactness for $p = 1$.

5.5. Borderline case for $n = 2$. The goal of this exercise is to prove that there is no continuous immersion of $W^{1,2}(\mathbb{R}^2)$ into $C^0(\mathbb{R}^2)$. As a counterexample, look at $u_\epsilon : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$u_\epsilon(z) = \begin{cases} \dfrac{\log |z|}{\log \epsilon} \text{ for} & \epsilon \leq |z| \leq 1 \\ 1 \text{ for} & |z| \leq \epsilon \\ 0 \text{ for} & |z| \geq 1 \end{cases}.$$

Prove that $u_\epsilon \in W^{1,2}(\mathbb{R}^2) \cap C^2(\mathbb{R}^2)$ and that there is no constant $C > 0$ such that for all $\epsilon > 0$,

$$\|u_\epsilon\|_{C^0(\mathbb{R}^2)} \leq C \, \|u_\epsilon\|_{W^{1,2}(\mathbb{R}^2)} \tag{1}$$

[1] Thank you to the student who spotted this hypothesis to be missing.

5.6. Give a counter example to show that the immersion $W^{1,2}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ is not compact.

Hint: For example start with u having compact support and construct a sequence by displacing its support by translation.

Please hand in your solutions for this sheet by Monday 04/04/2016.