5.1. Prove that the $L^1_{loc}(\mathbb{R})$ function $u : \mathbb{R} \to \mathbb{R} : x \mapsto |x|$, has a weak derivative in $L^1_{loc}(\mathbb{R})$.

Solution: We look at the sign function $v : \mathbb{R} \to \mathbb{R} : x \mapsto \mathrm{sgn}(x)$. Then we have for $x \neq 0$ that $u'(x) = v(x)$ and for $\varphi \in C_0^\infty(\mathbb{R})$, we have

$$\begin{aligned}\int_{\mathbb{R}} u(x)\varphi'(x)\ \mathrm{d}x &= \lim_{\epsilon\to 0^+} \int_{\{|x|>\epsilon\}} u(x)\varphi'(x)\ \mathrm{d}x \\ &= \lim_{\epsilon\to 0^+} \left(-\int_{\{|x|>\epsilon\}} v(x)\varphi(x)\ \mathrm{d}x + u(-\epsilon)\varphi(-\epsilon) - u(\epsilon)\varphi(\epsilon) \right) \\ &= -\int_{\mathbb{R}} v(x)\varphi(x)\ \mathrm{d}x\end{aligned}$$

where the second line follows from integration by part and the fact that φ has compact support. This shows that v is the weak derivative of u.

5.2. Weak derivative of K. Let $K := K_0$ be the fundamental solution of the Laplace operator, $n \geq 2$. Prove that the first strong derivative $\partial_i K$ of K, defined everywhere but the origin, is also the first weak derivative of K for $1 \leq i \leq n$.

N.B. Note that this is not true for the second derivatives, as K is not a weak solution for the Laplace equation, but still $\Delta K = 0$ everywhere but the origin.

Solution: We have already calculated in earlier exercises that

$$\partial_i K = \frac{x_i}{\omega_n |x|^n} \in L^1_{loc}(\mathbb{R}^n).$$

Therefore, for every function $\varphi \in C_0^\infty(\mathbb{R}^n)$, the functions $\mathbb{1}_{B_\epsilon(0)\setminus\{0\}}\varphi\partial_i K$ are bounded by $\mathbb{1}_{B_1(0)}|\partial_i K| \in L^1(\mathbb{R}^n)$, and converge point-wise almost everywhere to 0. Therefore by Lebesgue dominated convergence, we have

$$\lim_{\epsilon\to 0^+} \int_{B_\epsilon(0)} \varphi\partial_i K\ \mathrm{d}x = 0.$$

Similarly, we get $\lim_{\epsilon\to 0^+} \int_{\partial B_\epsilon(0)} \varphi K \nu^i\ \mathrm{d}S = 0$. Therefore, we get

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi\partial_i K\ \mathrm{d}x &= \lim_{\epsilon\to 0^+} \left(\int_{\mathbb{R}^n\setminus B_\epsilon(0)} \varphi\partial_i K\ \mathrm{d}x + \int_{B_\epsilon(0)} \varphi\partial_i K\ \mathrm{d}x \right) \\ &= \lim_{\epsilon\to 0^+} \left(-\int_{\mathbb{R}^n\setminus B_\epsilon(0)} \partial_i\varphi K\ \mathrm{d}x + \int_{\partial B_\epsilon(0)} \varphi K\nu^i\ \mathrm{d}S \right) \\ &= \int_{\mathbb{R}^n} \partial_i\varphi K\ \mathrm{d}x\end{aligned}$$

where the second line used integration by part.

5.3. Let $I = (a, b) \subset \mathbb{R}$ be a possibly unbounded open interval and let $1 \leq p \leq \infty$. Show that $u \in W^{1,p}(I)$ if and only if u is continuous, $u \in L^p(I)$[1] and there is $v \in L^p(I)$ such that

$$u(t) - u(s) = \int_s^t v(r)\ \mathrm{d}r$$

for all $t, s \in I$.

Solution: Take $u \in W^{1,p}(I)$. Let $v \in L^p(I)$ be its weak derivative. For this implication, we will repeatedly use Lebesgue's differentiation theorems from measure theory, which can be found for example in Theorem 7.10, 7.11 and 7.18 of Walter Rudin, Real and Complex Analysis [2].

Fix J a bounded open subinterval of I. Fix $s, t \in J$, $s < t$ which are Lebesgue points for $u \in L^1(J)$ [3]. This means that we consider almost all points s, t in J. Now consider the Lipschitz continuous functions $\varphi_\epsilon : I \to \mathbb{R}$ approximating $\mathbb{1}_{[s,t]}$ given by

$$\varphi_\epsilon(x) = \begin{cases} \frac{1}{\epsilon}(x - (s - \epsilon/2)) \text{ for} & s - \frac{\epsilon}{2} \leq x \leq s + \frac{\epsilon}{2} \\ 1 \text{ for} & s + \frac{\epsilon}{2} \leq x \leq t - \frac{\epsilon}{2} \\ \frac{1}{\epsilon}((t + \epsilon/2) - x) \text{ for} & t - \frac{\epsilon}{2} \leq x \leq t + \frac{\epsilon}{2} \\ 0 \text{ else} & \end{cases}$$

for $\epsilon > 0$ sufficiently small such that it is well defined and $\operatorname{supp} \varphi_\epsilon \subset J$. Then we have its weak derivative $\dot{\varphi}_\epsilon = \frac{1}{\epsilon}\left(\mathbb{1}_{(s-\frac{\epsilon}{2}, s+\frac{\epsilon}{2})} - \mathbb{1}_{(t-\frac{\epsilon}{2}, t+\frac{\epsilon}{2})}\right)$ and so by differentiation theorem, we have

$$\lim_{\epsilon \to 0^+} \left(-\int_I u(r)\dot{\varphi}_\epsilon(r)\ \mathrm{d}r\right) = u(t) - u(s)$$

as s, t were Lebesgue points. Moreover by dominated convergence theorem as $v \in L^1(J)$, we have

$$\lim_{\epsilon \to 0^+} \int_I v(r)\varphi_\epsilon(r)\ \mathrm{d}r = \int_s^t v(r)\ \mathrm{d}r.$$

[1] Thank you to the student who spotted this hypothesis to be missing.
[2] ISBN 9780070542341, 3rd edition, McGraw-Hill, 1987
[3] This follows from Hölder's inequality for $u \in L^p(J)$ and $1 \in L^q(J)$.

As $\overline{J}$ is compact, φ_ϵ will be in $W^{1,1}(J)$, and so can be approximated by functions $\psi_k \in C_0^\infty(I)$ with support in J in the $W^{1,1}(I)$ norm. Hence by definitions, we have

$$-\int_I u(r)\dot{\varphi}_\epsilon(r)\ \mathrm{d}r = \lim_{k\to\infty} -\int_I u(r)\dot{\psi}_k(r)\ \mathrm{d}r = \lim_{k\to\infty}\int_I v(r)\psi_k(r)\ \mathrm{d}r = \int_I v(r)\varphi_\epsilon(r)\ \mathrm{d}r.$$

By uniqueness of limits and the fact that J was arbitrary, we thereby have proven that for almost all $s, t \in I$, $s < t$,

$$u(t) - u(s) = \int_s^t v(r)\ \mathrm{d}r$$

Hence up to redefining u on a set of measure zero, we have

$$u(t) - u(s) = \int_s^t v(r)\ \mathrm{d}r$$

and so u is continuous by dominated convergence. Indeed, let $t \in I$ and pick $t_k \to t$ a convergent sequence. As t_k is then a bounded sequence there is J a bounded open subinterval of I containing all t_k. Now $v \in L^1(J)$, means that the sequence $v_k := v\mathbb{1}_{(\min(t_k,t),\max(t_k,t))}$ is dominated in $L^1(I)$ by $v\mathbb{1}_J$ and converges point-wise to 0 and so

$$\lim_{t\to\infty} u(t_k) = u(t) + \lim_{k\to\infty}\int_I v_k\ \mathrm{d}r = u(t).$$

Hence, u is continuous at every point $t \in I$.

Alternate solution: [4] We can also use J a bounded open subinterval of I and approximate u in $W^{1,p}(J)$, i.e. there is $u_k \in C_0^\infty(J)$ such that $u_k \to u$ and $u_k' \to v$ in $L^p(J)$ and the convergence is also almost everywhere. As J is bounded, u_k and u_k' also converges in $L^1(J)$. Therefore, we get as the fundamental theorem of calculus holds for $C_0^\infty(J)$ functions,

$$\int_s^t v(r)\ \mathrm{d}r = \lim_{k\to\infty}\int_s^t u_k'(r)\ \mathrm{d}r = \lim_{k\to\infty}(u_k(t) - u_k(s)) = u(t) - u(s)$$

for all $t, s \in J$ such that $u_k(t) \to u(t)$ and $u_k(s) \to u(s)$. Hence this is true for almost all $s, t \in J$. The rest of the conclusion still works in the same way.

For the converse, we will prove that v is the weak differential of u. Therefore, fix $\varphi \in C_0^\infty(I)$, J a bounded open subinterval of I containing $\operatorname{supp}\varphi$ and $t_0 \in J$. We calculate

$$\begin{aligned}\int_I u(r)\dot{\varphi}(r)\ \mathrm{d}r &= \int_I \left(u(t_0) + \int_{t_0}^r v(t)\ \mathrm{d}t\right)\dot{\varphi}(r)\ \mathrm{d}r\\ &= \int_I\int_{t_0}^r v(t)\ \mathrm{d}t\ \dot{\varphi}(r)\ \mathrm{d}r\\ &= -\int_I v(r)\varphi(r)\ \mathrm{d}r\end{aligned}$$

[4] Suggested to me by Francesco Palmurella

where we used the assumption on v in the first equality, $\dot{\varphi}$ having compact support in I in the second equality and the Lebesgue differentiation theorem[5] is used for the last equality. So $u \in W^{1,p}(I)$.

5.4. Embedding theorem for $n = 1$. Let $I = (a, b)$ be a bounded, open interval in $\mathbb{R}$ and $1 \leq p \leq \infty$. Prove that u is in the Hölder space $C^{0,1-\frac{1}{p}}(I)$ and that for $v \in L^p(I)$ the weak derivative of u, we have

$$\sup_{s,t \in I, t \neq s} \frac{|u(t) - u(s)|}{|t - s|^{1-\frac{1}{p}}} \leq \|v\|_{L^p(I)}$$

Deduce that the immersion $W^{1,p}(I) \to C^0(I)$ is compact for $p > 1$ and find a counterexample to compactness for $p = 1$.

Solution: For the first statement, uses the previous exercise to choose u continuous and

$$u(t) - u(s) = \int_s^t v(r) \, \mathrm{d}r.$$

Therefore, by Hölder inequality for $\frac{1}{p} + \frac{1}{q} = 1$, we get

$$|u(t) - u(s)| \leq \int_I |v(r)| \, \mathrm{d}r \leq \left(\int_s^t 1 \, \mathrm{d}r \right)^{1/q} \|v\|_{L^p(I)} \leq \|v\|_{L^p(I)} |t - s|^{1-\frac{1}{p}}.$$

which proves $u \in C^{0,1-\frac{1}{p}}(I)$.

For the second statement, for $p > 1$, we use Arzelà-Ascoli theorem which proves that $C^{0,\alpha}(I) \to C^0(I)$ is compact for $\alpha > 0$.

For the second statement, for $p = 1$, we take $I = (-1, 1)$ and look at

$$u_n(x) = \begin{cases} 1 & \text{for } \frac{1}{n} < x < 1 \\ nx & \text{for } -\frac{1}{n} \leq x \leq \frac{1}{n} \\ -1 & \text{for } -1 < x < -\frac{1}{n} \end{cases}.$$

Then the weak derivative is $\dot{u}_n = n\mathbb{1}_{(-\frac{1}{n},\frac{1}{n})}$. Therefore, we have $\|u_n\| \leq 2 + 1$ for all $n \in \mathbb{N}$ and $u_n \to u := -\mathbb{1}_{(-1,0)} + \mathbb{1}_{(0,1)}$ in $L^1(I)$. This proves that the inclusion cannot

[5] By 7.18, u is AC on J, so $u\varphi$ is also AC on J and therefore we have the integration by part formula by 7.18 and 7.12.

be compact, because a uniform convergent subsequence u_{k_j} would also converge in $L^1(I)$ due to boundedness of I and the only L^1 limit point is $u \notin C^0(I)$.

5.5. Borderline case for $n = 2$. The goal of this exercise is to prove that there is no continuous immersion of $W^{1,2}(\mathbb{R}^2)$ into $C^0(\mathbb{R}^2)$. As a counterexample, look at $u_\epsilon : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$u_\epsilon(z) = \begin{cases} \dfrac{\log|z|}{\log\epsilon} \text{ for} & \epsilon \le |z| \le 1 \\ 1 \text{ for} & |z| \le \epsilon \\ 0 \text{ for} & |z| \ge 1 \end{cases}.$$

Prove that $u_\epsilon \in W^{1,2}(\mathbb{R}^2) \cap C^2(\mathbb{R}^2)$ and that there is no constant $C > 0$ such that for all $\epsilon > 0$,

$$\|u_\epsilon\|_{C^0(\mathbb{R}^2)} \le C \|u_\epsilon\|_{W^{1,2}(\mathbb{R}^2)} \tag{1}$$

Solution: We calculate that

$$\partial_1 u_\epsilon(z) = \frac{x}{|z|^2 \log(\epsilon)}, \qquad \partial_2 u_\epsilon(z) = \frac{x}{|z|^2 \log(\epsilon)}$$

for $\epsilon < |z| < 1$, and that $\partial_i u(z) = 0$ for $|z| < \epsilon$ and $|z| > 1$. Hence, for $\varphi \in C_0^\infty(\mathbb{R}^2)$, we have

$$\int_{\mathbb{R}^2} u\partial_i\varphi \,\mathrm{d}x = \int_{|z|<\epsilon} u\partial_i\varphi \,\mathrm{d}x + \int_{\epsilon<|z|<1} u\partial_i\varphi \,\mathrm{d}x + \int_{|z|>1} u\partial_i\varphi \,\mathrm{d}x := I_1 + I_2 + I_3$$

where

$$\begin{aligned} I_1 &= \int_{|z|<\epsilon} \partial_i\varphi \,\mathrm{d}x = \int_{|z|=\epsilon} \varphi\nu_i \mathrm{d}S \\ I_2 &= \int_{\epsilon<|z|<1} u\partial_i\varphi \,\mathrm{d}x = -\int_{|z|=\epsilon} \varphi\nu_i \mathrm{d}S - \int_{\epsilon<|z|<1} \partial_i u\varphi \,\mathrm{d}x \\ I_3 &= 0 \end{aligned}$$

Hence, (as always when the function is continuous and differentiable everywhere but on a hypersurface) u has weak gardient $\nabla u_\epsilon(z) = \mathbb{1}_{\epsilon<|z|<1} \frac{z}{|z|^2 \log\epsilon^2}$ equal to the strong gradient where ever defined. As $0 \le u_\epsilon(z) \le \mathbb{1}_{B_1(0)}$, $u \in L^2(\mathbb{R}^2)$ and obviously $u \in C^0(\mathbb{R}^2)$. Furthermore,

$$\int_{R^2} |\nabla u_\epsilon(z)|^2 \,\mathrm{d}z = \int_{\epsilon<|z|<1} \frac{1}{|z|^2 \log(\epsilon)^2} \,\mathrm{d}z = 2\pi \int_\epsilon^1 \frac{1}{r\log(\epsilon)^2} \,\mathrm{d}r = \frac{2\pi}{|\log\epsilon|} \xrightarrow{\epsilon\to 0} 0.$$

Hence, $u_\epsilon \in W^{1,2}(\mathbb{R}^2) \cap C^0(\mathbb{R}^2)$. Also, $u_\epsilon|_{\partial B_r(0)} = 0$ for $r \geq 1$, so we also have for example $u_\epsilon \in W_0^{1,2}(B_2(0))$[6], thus by Poincaré inequality for $\Omega = B_2(0)$, we get $\|u_\epsilon\|_{L^2} \to 0$ as well. Hence, $\|u_\epsilon\|_{W^{1,2}(\mathbb{R}^2)} \to 0$, whereas $\|u_\epsilon\|_{C^0(\mathbb{R}^2)} = 1$, so there can be no $C > 0$ in (1).

5.6. Give a counter example to show that the immersion $W^{1,2}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ is not compact.

Hint: For example start with u having compact support and construct a sequence by displacing its support by translation.

Solution: Let $u \in W^{1,2}(\mathbb{R}^n)$ be a function with compact support and such that $\|u\|_{W^{1,2}} = 1$. Then fix a vector $v \in \mathbb{R}^n$ and define the sequence $u_k \in W^{1,2}(\mathbb{R}^n)$ by $u_k(x) := u(x + kv)$. Now $\|u_k\|_{W^{1,2}(\mathbb{R}^n)} = 1$, so if the inclusion were compact, we would have a subsequence u_{k_j} and a function $v \in L^2(\mathbb{R}^n)$ such that $u_{k_j} \xrightarrow{j\to\infty} v$. But then u_{k_j} also converges weakly to v in $L^2(\mathbb{R}^n)$. However, we will show that $u_k \rightharpoonup 0$ and so $v = 0$ in contradiction to $\|u_k\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)} > 0$ due to the translation invariance of the Lebesgue measure.

So take $\varphi \in C_0^\infty(\mathbb{R}^n)$ and note that the support u_k and φ will be disjoint for k sufficiently big. Hence

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} u_k \varphi \, \mathrm{d}x = 0$$

and so by density of $C_0^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, we have the same limit for $\varphi \in L^2(\mathbb{R}^n)$. All in all, this means that $u_k \rightharpoonup 0$.

[6] It is also in $W_0^{1,2}(B_1(0))$, but this is harder to see. This falls into the theorems involving the trace operator, which characterises $W_0^{1,2}(\Omega)$ for Ω with C^1 boundary.