6.1. Let $\Omega \subset \mathbb{R}^n$ be open and $u : \Omega \to \mathbb{R}$.

Prove that the following are equivalent:

(i) $u \in W^{1,\infty}_{loc}(\Omega)$.

(ii) u is locally Lipschitz.

Hint: For (i) implies (ii) use mollifiers $\rho_\delta(x) := \frac{1}{\delta^n}\rho(x/\delta)$ with $\operatorname{supp}\rho \subset B_1(0)$ and estimate the Lipschitz constant for $u_\delta = u * \rho_\delta$ as $\delta \to 0$. For (ii) implies (i), consider a fixed vector $\xi \in \mathbb{R}^n$ and define the difference quotient

$$u_j(x) := j\left[u(x + \frac{\xi}{j}) - u(x)\right].$$

Prove that there is $u^\xi \in L^\infty_{loc}$, such that a subsequence of u_j weakly converges to u^ξ in L^2_{loc}. Show that

$$\int_\Omega u^\xi \varphi = -\int_\Omega u \partial_\xi \varphi$$

for any $\varphi \in C_0^\infty(\Omega)$ by proving a similar equality for u_j and taking the limit.

Solution: <u>(i) $\Rightarrow$ (ii):</u> Let $u \in W^{1,\infty}_{loc}(\Omega)$. Then $u_\delta = \rho_\delta * u \in C_0^\infty(\mathbb{R}^n)$[1] converges to u almost everywhere on Ω. Also

$$|\nabla u_\delta(x)| \leq \|\nabla u\|_{L^\infty(B_\delta(x))} \int_{\mathbb{R}^n} \rho_\delta(y)\ \mathrm{d}y = \|\nabla u\|_{L^\infty(B_\delta(x))}$$

for $\delta > 0$ with $B_\delta(x) \subset \Omega$. Hence for $x \in \Omega$ where $u_\delta(x)$ converges to $u(x)$, let $\delta_0 > 0$ such that $\overline{B_{\delta_0}(x)} \subset \Omega$. Then for $x + \xi \in B_{\delta_0}(x)$ such that $u_\delta(x+\xi)$ converges to $u(x+\xi)$, we get

$$\begin{aligned}|u(x) - u(x+\xi)| &= \lim_{\delta\to 0} |u_\delta(x) - u_\delta(x+\xi)| = \lim_{\delta \to 0}\left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} u_\delta(x + s\xi)\ \mathrm{d}s\right| \\ &= \lim_{\delta\to 0}\left|\int_0^1 \nabla u(x+s\xi)\cdot\xi\ \mathrm{d}s\right| \leq \lim_{\delta\to 0}|\xi|\int_0^1 |\nabla u(x+s\xi)|\ \mathrm{d}s \\ &\leq \lim_{\delta \to 0}\|\nabla u_\delta\|_{L^\infty(B_\delta(x))}|\xi| \leq \|\nabla u\|_{L^\infty(B_{\delta_0}(x))}|\xi|\end{aligned}$$

and so $C_x := \|\nabla u\|_{L^\infty(\overline{B_{\delta_0}(x)})} < \infty$. As C_x is independent of ξ, we can change u on a set of measure zero on K and we get that $u|_{B_{\delta_0(x)}}$ is Lipschitz continuous on $B_{\delta_0}(x)$ with constant C_x.

[1] As usual, you extend u outside of Ω in $\mathbb{R}^n$ by definition of $W^{1,p}(\Omega)$ and then take this function for the convolution.

So far, we know how to make u Lipschitz in a neighbourhood of almost every point. Now we need to convince ourselves that we can make changes to u on a set of measure zero such that u is locally Lipschitz everywhere. For this use σ compactness of Ω. Indeed, fix a nested sequence of compact sets $\Omega_n \subset \Omega_{n+1}$ that exhausts [2] Ω and now for every $n \in \mathbb{N}$, we change u on a set of measure zero on Ω_n. We note that if $u|_{\Omega_n}$ is locally Lipschitz, we don't have to change u on Ω_n to make it locally Lipschitz in Ω_{n+1}. Now for the Lebesgue measure, full measure sets are also dense, hence for this dense set $\{x_\nu \in \Omega_n\}$, $B_{\delta_0(x_\nu)}(x_\nu)$ covers Ω_n. Thus finitely many cover, and for these finitely many balls, change u on a set of measure zero as before, to make $u|_{\Omega_n}$ locally Lipschitz. This proves that we can change u on a set of measure zero, to make it locally Lipschitz on Ω as the countable union of measure zero sets is of measure zero.

<u>(ii) $\Rightarrow$ (i):</u> Define u_j as in the hint where we extend u outside of Ω. Let $\Omega_n \subset \Omega_{n+1}$ an exhausting, increasing sequence of bounded open sets. Then the Ω_n can be chosen such that for $n \in \mathbb{N}$, there is $j \in \mathbb{N}$ such that $x + \xi/j \in \Omega_{n+1}$ for every $x \in \Omega_n$. Then there is a Lipschitz constant C_n for $\overline{\Omega}_{n+1}$ by finite covering property and so we have

$$|u_j(x)| \leq jC_n \frac{|\xi|}{j} = C_n |\xi|$$

for all $x \in \Omega_n$. So therefore, $\|u_j\|_{L^2(\Omega_n)} \leq C_n |\xi| \operatorname{vol}(\Omega_n)^{1/2}$. As $L^2(\Omega_n)$ is reflexive and separable, we have by Banach-Alaoglu theorem from FAI a weakly convergent subsequence $u_{j_k} \rightharpoonup u_n^\xi \in L^2(\Omega_n)$. Making this construction inductively with the previous subsequence, we get by uniqueness of weak limits, that $u_n^\xi|_{\Omega_n} = u_{n+1}^\xi|_{\Omega_n}$. Thus we can define the function $u^\xi : \Omega \to \mathbb{R}$ by $u^\xi|_{\Omega_n} := u_n^\xi$. As Ω_n is exhausting, the diagonal subsequence will weakly converge to u^ξ in $L^2_{loc}(\Omega)$.

Now to prove that $u^\xi \in L^\infty_{loc}(\Omega)$. This comes from the fact that every u_j defines a linear functional on L^1_{loc}, whose dual is L^∞_{loc} and the Banach-Steinhaus theorem from FAI. Namely, we have that for Ω_n as before, that for $v \in L^1(\Omega_n)$ that

$$\left| \int_{\Omega_n} u_j v \right| \leq C_n \|v\|_{L^1(\Omega_n)}$$

where C_n is a Lipschitz constant for $\overline{\Omega_{n+1}}$ and we have that $L_j v := \int_{\Omega_n} u_j \varphi$ is Cauchy on the dense subset $L^2(\Omega_n) \subset L^1(\Omega_n)$ [3]. Hence by Banach-Steinhaus, there is an operator $L \in L^1(\Omega_n)^*$ such that L_j strongly converges to L. This means in particular, that there is a function $w \in L^\infty(\Omega_n)$ such that $Lv = \int_{\Omega_n} wv$ for all $v \in L^1(\Omega_n)$. Due to strong convergence, we also have for every $v \in L^2(\Omega_n)$ that $Lv = \int_{\Omega_n} u^\xi v$. Hence,

[2] This means that $\bigcup_{i=1}^\infty \Omega_n = \Omega$.

[3] Density follows from the fact that $L^2(\Omega_n)$ already contains all the $C^0(\Omega_n)$ which form a dense subset.

w is also a weak limit of u_j on $L^2(\Omega_n)$. By uniqueness of weak limits, we have $u^\xi = w$ and so $u^\xi \in L^\infty(\Omega_n)$.

Let us prove that u^ξ is the weak derivative of u in the direction of ξ, namely for u_j we have the following for every $\varphi \in C_0^\infty(\mathbb{R}^n)$:

$$\begin{aligned}\int_{\mathbb{R}^n} j[u\left(x+\frac{\xi}{j}\right) - u(x)]\varphi(x) &= \int_{\mathbb{R}^n} ju\left(x+\frac{\xi}{j}\right)\varphi(x) - \int_{\mathbb{R}^n} ju(x)\varphi(x) \\ &= \int_{\mathbb{R}^n} ju(x)\varphi\left(x-\frac{\xi}{j}\right) - \int_{\mathbb{R}^n} ju(x)\varphi(x) \\ &= \int_{\mathbb{R}^n} u(x)[j(\varphi\left(x-\frac{\xi}{j}\right) - \varphi(x)]\end{aligned}$$

where we used invariance under translation of the Lebesgue measure in the second line. Therefore, as we take $j \to \infty$, by definition of u^ξ, we get

$$\int_{\mathbb{R}^n} u^\xi \varphi = -\int_{\mathbb{R}^n} u\partial_\xi \varphi.$$

6.2. Let $\Omega \subset \mathbb{R}^2$ be defined by

$$\begin{aligned}&\Omega := \Omega_0 \cup \bigcup_{m=0}^{\infty} \Omega_m \\ &\Omega_0 := \{(x,y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < \frac{1}{2}\} \\ &\Omega_m := \{(x,y) \in \mathbb{R}^2 : \frac{1}{2^{2m+1}} < x < \frac{1}{2^{2m}}, \frac{1}{2} \leq y < 1\}\end{aligned}$$

(a) Show that the embedding $W^{1,2}(\Omega) \to L^2(\Omega)$ is not compact.

(b) Show that $W^{1,2}(\Omega)$ is not a subset of $L^q(\Omega)$ for $q > 2$.

Solution:

(a) Consider the function u_m supported in Ω_m given by

$$u_m(x,y) = \begin{cases} 2^m & \text{for } (x,y) \in \Omega_m, \frac{3}{4} < y < 1 \\ 4(y-\frac{1}{2})2^m & \text{for } (x,y) \in \Omega_m, \frac{1}{2} < y < \frac{3}{4} \\ 0 & \text{else} \end{cases}$$

Then a short calculation shows that

$$\frac{1}{4} \leq \int_\Omega |u_m|^2 \, \mathrm{d}x \leq \frac{1}{2}, \qquad \int_\Omega |\nabla u_m|^2 \leq 4$$

and so $\{u_m\}$ is bounded in $W^{1,2}(\Omega)$. Assume now that a subsequence u_m would converge to $u \in L^2(\Omega)$. Then since u_m converges point wise almost everywhere to zero, we see by dominated convergence (Ω is bounded) that $u \equiv 0$. But this gives us a contradiction to $\frac{1}{4} \leq \int_\Omega |u_m|^2 \, \mathrm{d}x$.

(b) Let us define the function u_m as

$$u_m(x,y) = \begin{cases} \dfrac{2^m}{m} & \text{for } (x,y) \in \Omega_m, \dfrac{3}{4} < y < 1 \\ \dfrac{4}{m}(y - \dfrac{1}{2})2^m & \text{for } (x,y) \in \Omega_m, \dfrac{1}{2} < y < \dfrac{3}{4} \\ 0 & \text{else} \end{cases}$$

and $u = \sum_{m=1}^\infty u_m$. Then a similar calculation as in (a) shows that

$$\int_\Omega |u_m|^2 \, \mathrm{d}x \leq \frac{1}{2m^2}, \qquad \int_\Omega |\nabla u_m|^2 \leq \frac{4}{m^2} \qquad \int_\Omega |u_m|^q \, \mathrm{d}x \geq \frac{2^{(q-2)m}}{4m^q}$$

for all $q > 2$. Therfore,

$$\|u\|^2_{W^{1,2}(\Omega)} \leq \sum_{m=1}^\infty (\frac{1}{2} + 4)\frac{1}{m^2} < \infty$$

whereas

$$\|u\|^q_{L^q(\Omega)} \geq \frac{1}{4} \sum_{m=1}^\infty \frac{2^{(q-2)m}}{m^q} = \infty.$$

Hence $u \in W^{1,2}(\Omega) \setminus L^q(\Omega)$.

6.3. Show that $C^\infty(\overline{\Omega})$ is not dense in $W^{1,p}(\Omega)$ for $p \geq 1$ where:

(a) $\Omega = (-1, 0) \cup (0, 1)$.

(b) $\Omega := \{(x,y) \in \mathbb{R}^2 : |(x,y)| < 1\} \setminus \{(x,0) \in \mathbb{R}^2 : 0 \leq x < 1\}$.

Hint: For (b), prove that for $0 < \epsilon \leq 1$ and for any smooth function $\varphi : [-\epsilon, \epsilon] \to \mathbb{R}$, one has

$$\int_{-\epsilon}^0 |\varphi(t)| \, \mathrm{d}t + \int_0^\epsilon |1 - \varphi(t)| \, \mathrm{d}t + \int_{-\epsilon}^\epsilon |\varphi'(t)| \, \mathrm{d}t \geq \epsilon.$$

Then consider a function $u \in W^{1,p}(\Omega)$ which cannot be extended to a continuous function on $B_1(0)$ and find a contradiction once you try to approximate it by smooth functions.

Solution:

(a) Let $p \geq 1$. Consider the function $u := \mathbb{1}_{(0,1)} \in L^p(\Omega)$, and assume that there is a sequence $u_m \in C^\infty((\overline{\Omega}) = C^\infty((-1,1))$ converging to u in $W^{1,p}(\Omega)$. Then $u \in W^{1,p}((-1,1))$, but this is a contradiction to the fact that every element of $W^{1,p}((-1,1))$ has a continuous representative.

(b) To prove the suggested inequality in the hint, set $s := \sup_{(0,\epsilon)} \varphi$ and $r = \inf_{(-\epsilon,0)} \varphi$. We always have $r \leq s$. Let us only treat the case where $0 \leq r \leq s \leq 1$, for the other cases are similar. Then we get the following estimates

$$\int_{-\epsilon}^{0} |\varphi| \, \mathrm{d}t \geq r\epsilon, \qquad \int_{0}^{\epsilon} |\varphi| \, \mathrm{d}t \geq (1-s)\epsilon, \qquad \text{and} \int_{-\epsilon}^{\epsilon} |\dot{\varphi}| \, \mathrm{d}t \geq s - r.$$

As $\epsilon \leq 1$, we conclude as $(s-r)(1-\epsilon) + \epsilon \geq \epsilon$. This inequality basically tells us that a smooth function cannot jump from 0 to 1 without recking up some derivative.

Let $p \geq 1$. Now consider the cube Q centred at $x = (\frac{1}{2}, 0)$ of side length $2\epsilon < \frac{1}{4}$. Then $Q \cap \Omega$ has two connected components Q_+ and Q_-. Fix u to be 1 on Q_+ and 0 on Q_-. Then u can be extended to a function (even smooth by cutting off) on Ω of class $W^{1,p}(\Omega)$. Now assume there is a sequence of functions $\varphi_n \in C^\infty(\overline{\Omega})$ converging to u in $W^{1,p}(\Omega)$. Then we apply the above inequality to every function $\varphi_{n,x}(t) := \varphi_n(x,t)$ for $x \in (-\epsilon, \epsilon)$ and $n \in \mathbb{N}$, to get by Fubini

$$\int_Q |u - \varphi_n| + \int_Q |\nabla u - \nabla u_n| \geq 2\epsilon^2.$$

This means that $\|u - \varphi_n\|_{W^{1,1}(\Omega)} \geq 2\epsilon^2$ for all $n \in \mathbb{N}$. Therefore, as Ω is bounded, there is a constant$C(p) > 0$ by Hölder, such that

$$C(p) \|u - \varphi_n\|_{W^{1,p}(\Omega)} \geq \|u - \varphi_n\|_{W^{1,1}(\Omega)} \geq 2\epsilon^2.$$

This is a contradiction to φ_n converging to u in $W^{1,p}(\Omega)$.

6.4. Let $\Omega \subset \mathbb{R}^n$ be open. Let $u_n \in W^{k,p}(\Omega)$ be a Cauchy sequence with $u_n \to u$ in $L^p(\Omega)$. Prove that $u \in W^{k,p}(\Omega)$ and that $u_n \to u$ in $W^{k,p}(\Omega)$.

Solution: As $W^{1,p}(\Omega)$ is a Banach space, we have that $u_n \to v$ in $W^{k,p}(\Omega)$ where $v \in W^{k,p}(\Omega)$. As $\|w\|_{L^p(\Omega)} \leq \|w\|_{W^{k,p}(\Omega)}$ for all $w \in W^{k,p}(\Omega)$, $u_n \to v$ in $L^p(\Omega)$ as well. So by uniqueness of limits, $u = v$.

6.5. Let $\Omega = \mathbb{R}^n$, $p \geq 2$ and $u : \mathbb{R}^n \to \mathbb{R}$ in $W^{2,p}(\Omega)$.

(a) For $n = 1$, prove that

$$\int_{\mathbb{R}} |u'|^p \leq C(p) \int_{\mathbb{R}} |uu''|^{\frac{p}{2}}$$

(b) Prove that

$$\|u\|_{W^{1,p}(\Omega)} \leq C(n,p) \|u\|_{L^p(\Omega)}^{1/2} \|u\|_{W^{2,p}(\Omega)}^{1/2}$$

(c) Prove that

$$\|u\|_{W^{1,p}(\Omega)} \leq C(n,p) \|u\|_{L^\infty(\Omega)}^{1/2} \|u\|_{W^{2,\frac{p}{2}}(\Omega)}^{1/2}$$

Hint: For (a) start with a compactly supported smooth function u and consider $v := u' |u'|^{p-2}$. Then use the integration by part formula for $w = uv$ and the generalised Hölder inequality.

Solution:

(a) It is by density enough to prove the inequality for $u \in C_0^\infty(\mathbb{R})$. Take $v = u' |u'|^{p-2}$, then this is a function in $C_c^1(\mathbb{R})$ and its derivative is $v' = (p-1) |u'|^{p-2} u''$. Then for $w = uv$, we have $w' = |u'|^p + (p-1)uu'' |u'|^{p-2}$. Hence, by integration by part, we have due to compact support

$$\int_{\mathbb{R}} |u'|^p = (p-1) \int_{\mathbb{R}} uu'' |u'|^{p-2} \leq (p-1)(\int_{\mathbb{R}} |u''u|^{\frac{p}{2}})^{\frac{2}{p}}(\int_{\mathbb{R}} |u'|^{(p-2)\frac{p}{p-2}})^{\frac{p-2}{p}}$$

where we used Hölder with exponents $\frac{p}{2}$ and $\frac{p}{p-2}$. Hence by dividing by $(\int_{\mathbb{R}} |u'|^{(p-2)\frac{p}{p-2}})^{\frac{p-2}{p}}$, we get

$$(\int_{\mathbb{R}} |u'|^p)^{\frac{2}{p}} \leq (p-1)(\int_{\mathbb{R}} |u''u|^{\frac{p}{2}})^{\frac{2}{p}} \Leftrightarrow \int_{\mathbb{R}} |u'|^p \leq (p-1)^{\frac{p}{2}} \int_{\mathbb{R}} |uu''|^{\frac{p}{2}}$$

(b) Again fix $u \in C_0^\infty(\Omega)$. We have

$$\|u\|_{L^p(\Omega)} = \|u\|_{L^p(\Omega)}^{\frac{1}{2}} \|u\|_{L^p(\Omega)}^{\frac{1}{2}} \leq \|u\|_{L^p(\Omega)}^{1/2} \|u\|_{W^{2,p}(\Omega)}^{1/2}$$

By (a) for $i = 1, \dots, n$, we have for $x \in \Omega$ with $x_i = 0$, that

$$\int_{\mathbb{R}} |\partial_i u(x+te_i)|^p \, \mathrm{d}t \leq C(p) \int_{\mathbb{R}} \left| u(x+te_i)\partial_i^2 u(x+te_i) \right|^{\frac{p}{2}} \, \mathrm{d}t$$

Now integrating over all the other components, we get

$$\int_{\Omega} |\partial_i u|^p \leq C(p) \int_{\Omega} \left| u\partial_i^2 u \right|^{\frac{p}{2}} \leq C(p)(\int_{\Omega} |u|^p)^{\frac{1}{2}}(\int_{\Omega} \left|\partial_i^2 u\right|^p)^{\frac{1}{2}}$$

where we used Hölder with exponents 2 and 2. And hence taking the p^{th} root and summing up all the estimates, we get

$$\|u\|_{W^{1,p}(\Omega)} \leq C(n,p)\,\|u\|_{L^p(\Omega)}^{1/2}\,\|u\|_{W^{2,p}(\Omega)}^{1/2}$$

(c) As before in (b), we get

$$\int_\Omega |\partial_i u|^p \leq C(p)\int_\Omega \left|u\partial_i^2 u\right|^{\frac{p}{2}} \leq C(p)\,\|u\|_{L^\infty(\Omega)}^{\frac{p}{2}}\int_\Omega \left|\partial_i^2 u\right|^{\frac{p}{2}}$$

And hence taking the p^{th} root and summing up all the estimates, we get

$$\|u\|_{W^{1,p}(\Omega)} \leq C(n,p)\,\|u\|_{L^\infty(\Omega)}^{1/2}\,\|u\|_{W^{2,\frac{p}{2}}(\Omega)}^{1/2}$$