7.1. Special cases of Gagliardo-Nirenberg Use exercise 6.1, to derive the following inequality

$$\left\|\partial^j u\right\|_{L^q} \leq C \left\|u\right\|_{L^r}^{1-\lambda} \left\|\partial^k u\right\|_{L^p}^{\lambda}$$

for all $u \in C^\infty(\mathbb{R}^n)$ and where $C > 0$ only depends on $j, k, n, q, p, r, \lambda$ for the case where

(a) $p \geq 2, \lambda = \frac{j}{k}, j \leq k, q = p = r$.

(b) $p \geq 2, \lambda = \frac{j}{k}, j \leq k, q = \frac{kp}{j}, r = \infty, jq = kp > n$.

7.2. Poincaré inequality Let $1 \leq p \leq \infty$ and $\Omega \subset \mathbb{R}^n$ be a bounded open subset with C^1 boundary. Then there is $C := C(\Omega, p) > 0$ such that for all $u \in W^{1,p}(\Omega)$, we have

$$\left\|u - \overline{u}\right\|_{L^p(\Omega)} \leq C \left\|\nabla u\right\|_{L^p(\Omega)} \tag{1}$$

where $\overline{u} := \frac{1}{|\Omega|} \int_\Omega u(y) \, \mathrm{d}y$.

Hint: Assume by contradiction that there is a counter-example u_k for every $C = k \in \mathbb{N}$ in (1). Then subtract the average and renormalise in L^p, to get a sequence v_k. Now use Rellich-Kondrachov compactness result to get a contradiction.

7.3. Explosion of the constant in 7.2 Let $\Omega_k \subset \mathbb{R}^2$ be the domain of two squares connected with a small bridge[1]. In formulae,

$$\Omega_k := [-3, -1] \times [-1, 1] \cup [1, 3] \times [-1, 1] \cup [-1, 1] \times [0, \frac{1}{k}].$$

Check that $\lim_{k \to \infty} C_k = \infty$ where $C_k := C(\Omega_k, p)$ in (1).

7.4. Weak solutions of $\Delta u = \partial_j f$ Let $u, f \in L^1(\mathbb{R}^n)$ have compact support. Show that u is a weak solution of

$$\Delta u = \partial_j f \tag{2}$$

if and only if $u = \partial_j K * f$ where K is the fundamental solution of the Laplace operator.

[1] We write down a Lipschitz domain for ease of notation. One can smoothen the corners to get a C^1 domain as required in 7.2 or one can assume to know that Rellich-Kondrachov theorem also holds for Lipschitz domains which is true, but not proven in the lecture course.

Hint: Recall that $\partial_j K(x) = \frac{x_j}{\omega_n |x|^n}$ and exercise 5.2.

7.5. Subtle difference or maybe not. Prove that $W^{1,p}(\mathbb{R}^n) = W_0^{1,p}(\mathbb{R}^n)$, where $W_0^{1,p}(\mathbb{R}^n)$ is the closure of $C_0^\infty(\mathbb{R}^n)$ in $W^{1,p}(\mathbb{R}^n)$.

7.6. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Let $u \in W^{k,p}(\Omega)$ and suppose that

$$\partial^\alpha u|_{\partial\Omega} = 0,$$

for every multi-index α of order $|\alpha| \leq k-1$. Define $u : \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde{u}(x) := \begin{cases} u(x) & \text{for } x \in \Omega \\ 0 & \text{for } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Prove that $\tilde{u} \in W^{k,p}(\mathbb{R}^n)$.

Hint: For $|\alpha| \leq k$, define $\tilde{u}_\alpha : \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde{u}_\alpha(x) = \begin{cases} \partial^\alpha u(x) & \text{for } x \in \Omega \\ 0 & \text{for } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Prove that $\tilde{u}_\alpha$ is the weak derivative of u associated with the multi-index α.

Please hand in your solutions for this sheet by Monday 18/04/2016.