7.1. Special cases of Gagliardo-Nirenberg Use exercise 6.5, to derive the following inequality

$$\left\|\partial^j u\right\|_{L^q} \leq C\,\|u\|_{L^r}^{1-\lambda}\left\|\partial^k u\right\|_{L^p}^{\lambda} \tag{1}$$

for all $u \in C^\infty(\mathbb{R}^n)$ and where $C > 0$ only depends on $j, k, n, q, p, r, \lambda$ for the case where

(a) $p \geq 2, \lambda = \frac{j}{k}, 0 < j < k, q = p = r.$

(b) $p \geq 2, \lambda = \frac{j}{k}, 0 < j < k, q = \frac{kp}{j}, r = \infty, jq = kp > n.$

Hint: For (a), use induction on k. Apply the induction hypothesis on functions of $\partial^1 u$ and derive an inequality for $\|\partial^1 u\|$ involving only terms you want to keep.

For (b), convince yourself that (1) holds with $\tilde{q} \geq 2$ and $\tilde{p}, \tilde{r}$ such that $\frac{1}{2\tilde{r}} + \frac{1}{2\tilde{p}} = \frac{1}{\tilde{q}}$, $k = 2$, $j = 1$. Then apply this with $\tilde{q} = \frac{kp}{k-1}, \tilde{r} = \frac{kp}{k-2}$ and $\tilde{p} = p$ to functions of $\partial^{k-2}u$. Again get rid of unwanted terms, by using the induction hypothesis for some special p^* to prove the case $j = k-1$. Now prove it for all other j.

Solution:

(a) We prove this by induction. The case $j = 1$, $k = 2$ was proven in Exercise 6.5. Now for $k \geq 3$, we will assume the inequality (1) with constant $C_{J,K}$ for $2 \leq K \leq k-1$ and $0 < J < K$. Fix $0 < j < k-1$. Apply the inequality (1) with $K = k-1$ and $J = j$ on the functions of $\partial^1 u$ to get

$$\left\|\partial^{j+1} u\right\|_{L^p} \leq C_{j,k-1}\left\|\partial^1 u\right\|_{L^p}^{1-\frac{j}{k-1}}\left\|\partial^k u\right\|_{L^p}^{\frac{j}{k-1}}$$

Also for $K = j+1$ and $J = 1$ to u, and get

$$\left\|\partial^1 u\right\|_{L^p} \leq C_{1,j+1}\|u\|_{L^p}^{1-\frac{1}{j+1}}\left\|\partial^{j+1} u\right\|_{L^p}^{\frac{1}{j+1}}.$$

Plugging the latter in the former on the righthand side, we get

$$\left\|\partial^{j+1} u\right\|_{L^p} \leq C_{j,k-1}(C_{1,j+1}\|u\|_{L^p}^{1-\frac{1}{j+1}}\left\|\partial^{j+1} u\right\|_{L^p}^{\frac{1}{j+1}})^{1-\frac{j}{k-1}}\left\|\partial^k u\right\|_{L^p}^{\frac{j}{k-1}}$$

$$\Rightarrow \left\|\partial^{j+1} u\right\|_{L^p}^{\frac{kj}{(j+1)(k-1)}} \leq C_{j,k-1}C_{1,j+1}\|u\|_{L^p}^{(1-\frac{1}{j+1})(1-\frac{j}{k-1})}\left\|\partial^k u\right\|_{L^p}^{\frac{j}{k-1}}$$

which implies (1) for $J = j+1$, $K = k$ and constant $C_{j+1,k} := (C_{j,k-1}C_{1,j+1})^{\frac{(k-1)(j+1)}{kj}}$. The only case missing is $J = 1$ and $K = k$. For this we combine (1) for $K = k-1$, $J = 1$ and $K = k$ and $J = k-1$.

$$\begin{aligned}\left\|\partial^1 u\right\|_{L^p} &\leq C_{1,k-1}\|u\|_{L^p}^{1-\frac{1}{k-1}}\left\|\partial^{k-1} u\right\|_{L^p}^{\frac{1}{k-1}}\\ &\leq C_{1,k-1}\|u\|_{L^p}^{1-\frac{1}{k-1}}(C_{k-1,k}\|u\|_{L^p}^{1-\frac{k-1}{k}}\left\|\partial^k u\right\|_{L^p}^{\frac{k-1}{k}})^{\frac{1}{k-1}}\end{aligned}$$

which is (1) with $C_{1,k} := C_{1,k-1} C_{k-1,k}^{\frac{1}{k-1}}$. This proves (a).

(b) We prove this by induction. The case $j = 1$, $k = 2$ was proven in Exercise 6.5. Now for $k \geq 3$, we will assume the inequality (1) for $2 \leq K \leq k-1$, $p^* \geq 2$ and $0 < J < K$. Start with the case $J = k-1$. By the same reasoning as in 6.5, we have the inequality (1) with $\tilde{q} \geq 2$ and $\tilde{p}, \tilde{r}$ such that $\frac{1}{2\tilde{r}} + \frac{1}{2\tilde{p}} = \frac{1}{\tilde{q}}$. This inequality with $\tilde{q} = \frac{kp}{k-1}, \tilde{r} = \frac{kp}{k-2}$ and $\tilde{p} = p$, applied to the functions of $\partial^{k-2} u$ reads

$$\left\|\partial^{k-1} u\right\|_{L^{\frac{kp}{k-1}}} \leq C_1 \left\|\partial^{k-2} u\right\|_{L^{\frac{kp}{k-2}}}^{\frac{1}{2}} \left\|\partial^k u\right\|_{L^p}^{\frac{1}{2}}.$$

Now to get rid of the term $\left\|\partial^{k-2} u\right\|_{L^{\frac{kp}{k-2}}}^{\frac{1}{2}}$, we write $\frac{kp}{k-2} = \frac{kp}{k-1}\frac{k-1}{k-2} = p^* \frac{K}{J}$ and so by induction hypothesis, we have

$$\left\|\partial^{k-2} u\right\|_{L^{\frac{kp}{k-2}}} \leq C_2 \left\|\partial^{k-1} u\right\|_{L^{\frac{kp}{k-1}}}^{\frac{k-2}{k-1}} \|u\|_{L^\infty}^{\frac{1}{k-1}}$$

Putting the latter into the former will again yield (1) as

$$\left\|\partial^{k-1} u\right\|_{L^{\frac{kp}{k-1}}} \leq C_1^{\frac{j}{k}} C_2^{\frac{j}{2k}} \|u\|_{L^\infty}^{1-\frac{k-1}{k}} \left\|\partial^k u\right\|_{L^p}^{\frac{k-1}{k}}.$$

For $0 < j < k-1$, we will use the decomposition $\frac{kp}{j} = \frac{kp}{k-1}\frac{k-1}{j} = p^* \frac{K}{J}$ together with the induction hypothesis and the case $J = k-1$ which we already established to get

$$\begin{aligned}
\left\|\partial^j u\right\|_{L^{\frac{kp}{j}}} &\leq C \left\|\partial^{k-1} u\right\|_{L^{\frac{kp}{k-1}}}^{\frac{j}{k-1}} \|u\|_{L^\infty}^{1-\frac{j}{k-1}} \\
&\leq C' (\|u\|_{L^\infty}^{1-\frac{k-1}{k}} \left\|\partial^k u\right\|_{L^p}^{\frac{k-1}{k}})^{\frac{j}{k-1}} \|u\|_{L^\infty}^{1-\frac{j}{k-1}} \\
&\leq C' \left\|\partial^k u\right\|_{L^p}^{\frac{j}{k}} \|u\|_{L^\infty}^{1-\frac{j}{k}}.
\end{aligned}$$

This finishes the proof of (b).

7.2. Poincaré inequality Let $1 \leq p \leq \infty$ and $\Omega \subset \mathbb{R}^n$ be a bounded open subset with C^1 boundary. Then there is $C := C(\Omega, p) > 0$ such that for all $u \in W^{1,p}(\Omega)$, we have

$$\|u - \overline{u}\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)} \tag{2}$$

where $\overline{u} := \frac{1}{|\Omega|} \int_\Omega u(y) \, dy$.

Hint: Assume by contradiction that there is a counter-example u_k for every $C = k \in \mathbb{N}$ in (2). Then subtract the average and renormalise in L^p, to get a sequence v_k. Now use Rellich-Kondrachov compactness result to get a contradiction.

Solution: As in the hint, assume there are $u_k \in W^{1,p}(\Omega)$ such that

$$\|u_k - \overline{u_k}\|_{L^p(\Omega)} \geq k \|\nabla u_k\|_{L^p(\Omega)}.$$

Then define

$$v_k := \frac{u_k - \overline{u_k}}{\|u_k - \overline{u_k}\|_{L^p(\Omega)}}$$

and observe that

$$\overline{v_k} = 0, \qquad \|\nabla v_k\|_{L^p(\Omega)} = 1 \qquad \text{and} \qquad \|\nabla v_k\|_{L^p(\Omega)} \leq \frac{1}{k}. \tag{3}$$

So the sequence v_k is bounded in $W^{1,p}(\Omega)$. Thus by Rellich-Kondrachov compactness, there is a subsequence v_{k_j} of v_k and a function $v \in L^p(\Omega)$, such that $v_{k_j} \to v$ in $L^p(\Omega)$ for $j \to \infty$. Thus in particular, we have

$$\overline{v} = 0 \qquad \text{and} \qquad \|v\|_{L^p(\Omega)} = 1.$$

Furthermore, we have for $\varphi \in C_0^\infty(\Omega)$ that for $i = 1, \ldots, n$

$$\int_\Omega v \partial_i \varphi = \lim_{j\to\infty} \int_\Omega v_{k_j} \partial_i \varphi = -\lim_{j\to\infty} \int_\Omega \partial_i(v_{k_j}) \varphi = 0$$

where we used a bit everywhere that Ω is bounded plus dominated convergence and in the last equality we used (3). Thus we have that $\nabla v = 0$, therefore we know that v is constant almost everywhere. This constant has to be zero as $\overline{v} = 0$ which contradicts $\|v\|_{L^p(\Omega)} = 1$.

7.3. Explosion of the constant in 7.2 Let $\Omega_k \subset \mathbb{R}^2$ be the domain of two squares connected with a small bridge[1]. In formulae,

$$\Omega_k := [-3, -1] \times [-1, 1] \cup [1, 3] \times [-1, 1] \cup [-1, 1] \times [0, \frac{1}{k}].$$

Check that $\lim_{k\to\infty} C_k = \infty$ where $C_k := C(\Omega_k, p)$ in (2).

Solution: For any large k, we wish to find a function $u_k \in W^{1,p}(\Omega_k)$ which forces the constant C_k of the inequality to be large. As k increases, the domain varies only by the bridge, which becomes increasing narrow. We exploit this to construct our functions u_k. They will concentrate their derivatives on the shrinking bridge taking

[1] We write down a Lipschitz domain for ease of notation. One can smoothen the corners to get a C^1 domain as required in 7.2 or one can assume to know that Rellich-Kondrachov theorem also holds for Lipschitz domains which is true, but has not proven in the lecture course.

constant values on the two large squares. Thus they will deviate significantly from the average. Indeed, set $u : \Omega_k \to \mathbb{R}$ to be

$$u_k(x, y) = \begin{cases} -1 \text{ for} & x \leq -1 \\ 1 \text{ for} & x \geq 1 \\ x \text{ for} & -1 \leq x \leq 1 \end{cases}$$

Thus, u_k is continuous on Ω_k, and by symmetry $\bar{u} = 0$. Let $\mathbb{1}_B$ be the characteristic function of the bridge. Then

$$\nabla u(x, y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbb{1}_B.$$

Moreover,

$$\|\nabla u\|^p_{L^p(\Omega_k)} = \int_{[-1,1]\times[0,\frac{1}{k}]} 1 \, \mathrm{d}x\mathrm{d}y = \frac{1}{k}$$

while at the same time

$$\|u - \overline{u}\|^p_{L^p(\Omega_k)} = \int_{\Omega_k} |u|^p \geq 8$$

Combining these two facts with the inequality (2), we see that $8 \leq C_k^p \frac{1}{k}$. Therefore, $\lim_{k\to\infty} C_k = \infty$.

7.4. Weak solutions of $\Delta u = \partial_j f$ Let $u, f \in L^1(\mathbb{R}^n)$ have compact support. Show that u is a weak solution of

$$\Delta u = \partial_j f \tag{4}$$

if and only if $u = \partial_j K * f$ where K is the fundamental solution of the Laplace operator.

Hint: Recall that $\partial_j K(x) = \frac{x_j}{\omega_n |x|^n}$ and exercise 5.2.

Solution: Start with u being a weak solution of our equation (4). This means that for every $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} u(y)\Delta\varphi(y) \, \mathrm{d}y = -\int_{\mathbb{R}^n} f(y)\frac{\partial}{\partial y^j}\varphi(y) \, \mathrm{d}y.$$

Plugging in the function $\rho_\delta(x - y)$ where ρ_δ is a mollifying kernel, we get

$$\Delta(\rho_\delta * u) = \partial_j(\rho_\delta * f).$$

where the minus disappears, as we differentiate with respect to y. Now taking the convolution product with K on both sides and recalling that the fundamental solution has the property $K * \Delta\varphi = \varphi$ for $\varphi \in C_0^\infty(\mathbb{R}^n)$, we get

$$\rho_\delta * u = K * \partial_j(\rho_\delta * f)$$

At this point we recall from Exercise 5.2, that $\partial_j K$ is the weak derivative of K. Now as we can distribute the derivatives over the factors as we want and as $*$ is commutative, we get

$$\rho_\delta * u = \rho_\delta * (\partial_j K * f).$$

As K_j is in $L^1_{loc}(\mathbb{R}^n)$, $\partial_j K * f \in L^1_{loc}(\mathbb{R}^n)$ and so as $\delta \to 0$, we have

$$u = \partial_j K * f$$

almost everywhere, so they agree as $L^1(\mathbb{R}^n)$ functions.

For the converse, assume $u = \partial_j K * f$. Recall from exercise 5.2, that and that $\partial_j K(x) = -(\partial_j K)(-x)$. Thus, we have for $\varphi \in C_0^\infty(\mathbb{R}^n)$, that

$$\begin{aligned}\int_{\mathbb{R}^n} u\Delta\varphi &= \int_{\mathbb{R}^n} (\partial_j K * f)\Delta\varphi \\ &= -\int_{\mathbb{R}^n} f(\partial_j K * \Delta\varphi) \\ &= -\int_{\mathbb{R}^n} f * (K * \Delta\partial_j\varphi) \\ &= -\int_{\mathbb{R}^n} f\partial_j\varphi.\end{aligned}$$

which is exactly the statement that u is weak solution of our equation (4).

7.5. Subtle difference or maybe not. Prove that $W^{1,p}(\mathbb{R}^n) = W_0^{1,p}(\mathbb{R}^n)$, where $W_0^{1,p}(\mathbb{R}^n)$ is the closure of $C_0^\infty(\mathbb{R}^n)$ in $W^{1,p}(\mathbb{R}^n)$.

Solution: Let $u \in W^{1,p}(\mathbb{R}^n)$ and let us try to approximate this function by functions of $C_0^\infty(\mathbb{R}^n)$. Fix $\epsilon > 0$. By a result of the course, we already have for ρ_δ a mollifying kernel, for $\delta > 0$ sufficiently small, we have

$$\|u - \rho_\delta * u\|_{W^{1,p}(\mathbb{R}^n)} \leq \frac{\epsilon}{4}$$

Now take a cut-off function $\beta \in C^\infty(\mathbb{R}^n)$ such that

$$\begin{aligned}&\beta \equiv 1 \text{ on } |x| \leq 1, \qquad \beta \equiv 0 \text{ on } |x| \geq 2, \\ &|\nabla\beta| \leq 2 \text{ and } 0 \leq \beta \leq 1 \text{ on } 1 \leq |x| \leq 2.\end{aligned}$$

Set $\beta_R(x) := \beta(R^{-1}x)$, $R > 0$. Therefore,

$$\beta_R \equiv 1 \text{ on } |x| \le R, \qquad \beta_R \equiv 0 \text{ on } |x| \ge 2R,$$
$$|\nabla \beta_R| \le 2R^{-1} \text{ and } 0 \le \beta_R \le 1 \text{ on } R \le |x| \le 2R.$$

Our approximating function will have the form $\beta_R(\rho_\delta * u) \in C_0^\infty(\mathbb{R}^n)$. For $R > 2$ big enough, we will have

$$\|\rho_\delta * u\|_{W^{1,p}(\mathbb{R}^n \setminus B_R(0))} \le \frac{\epsilon}{4}.$$

Hence,

$$\begin{aligned}
\|\rho_\delta * u - \beta_R(\rho_\delta * u)\|_{W^{1,p}(\mathbb{R}^n)} &= \|\rho_\delta * u\|_{W^{1,p}(\mathbb{R}^n \setminus B_{2R}(0))} \\
&\quad + \|\rho_\delta * u - \beta_R(\rho_\delta * u)\|_{W^{1,p}(B_{2R}(0) \setminus B_R(0))} \\
&\le \|\rho_\delta * u\|_{W^{1,p}(\mathbb{R}^n \setminus B_{2R}(0))} \\
&\quad + \|\rho_\delta * u - \beta_R(\rho_\delta * u)\|_{L^p(B_{2R}(0) \setminus B_R(0))} \\
&\quad + \|-(\nabla \beta_R)\rho_\delta * u\|_{L^p(B_{2R}(0) \setminus B_R(0))} \\
&\quad + \|(1 - \beta_R)\nabla(\rho_\delta * u)\|_{L^p(B_{2R}(0) \setminus B_R(0))} \\
&\le (2 + 2R^{-1})\frac{\epsilon}{4} \le \frac{3\epsilon}{4}
\end{aligned}$$

Thus all in all, we get

$$\|u - \beta_R(\rho_\delta * u)\|_{W^{1,p}(\mathbb{R}^n)} \le \epsilon.$$

Therefore, as $\epsilon > 0$ was arbitrary, we have $W_0^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$.

7.6. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Let $u \in W^{k,p}(\Omega)$ and suppose that

$$\partial^\alpha u|_{\partial\Omega} = 0,$$

for every multi-index α of order $|\alpha| \le k - 1$. Define $u : \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde{u}(x) := \begin{cases} u(x) & \text{for } x \in \Omega \\ 0 & \text{for } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Prove that $\tilde{u} \in W^{k,p}(\mathbb{R}^n)$.

Hint: For $|\alpha| \le k$, define $\tilde{u}_\alpha : \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde{u}_\alpha(x) = \begin{cases} \partial^\alpha u(x) & \text{for } x \in \Omega \\ 0 & \text{for } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Prove that $\tilde{u}_\alpha$ is the weak derivative of u associated with the multi-index α.

Solution: We want to prove that $\tilde{u}_\alpha$ is the weak derivative of index α of $\tilde{u}$. This means that for $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$(-1)^{|\alpha|} \int_{\mathbb{R}^n} \tilde{u} \partial^\alpha \varphi = \int_{\mathbb{R}^n} \tilde{u}_\alpha \varphi.$$

In other words, we need to prove that for $\varphi \in C^\infty(\overline{\Omega})$

$$(-1)^{|\alpha|} \int_\Omega u \partial^\alpha \varphi = \int_\Omega \partial^\alpha u \varphi.$$

We will prove this statement by induction on $k \in \mathbb{N}$, the number of derivatives of u.

For $k = 1$, the result was already established in class.

Fix $k \geq 2$. Now assume the induction hypothesis to hold for all $u \in W^{j,p}(\Omega)$ with $\partial^\alpha u|_{\partial\Omega} = 0$ for $|\alpha| \leq j-1$ and $1 \leq j \leq k$. We fix a multi-index α with $|\alpha| \leq k$ and m a non-zero component of α. Then for $u \in W^{1,p}(\Omega)$ with $\partial^\alpha u|_{\partial\Omega} = 0$ for $|\alpha| \leq k-1$, we have that $\partial^{\alpha - e_m} u \in W^{k,p}(\Omega)$ with $\partial^{\alpha-e_m} u|_{\partial\Omega} = 0$ and that $u \in W^{k-1,p}(\Omega)$ has $\partial^\beta u|_{\partial\Omega} = 0$ for all multi-index β with $\beta \leq (k-1)-1$. Hence we get by induction assumption, we get

$$(-1)^{|\alpha|} \int_\Omega u \partial^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega u \partial^{\alpha - e_m}(\partial_m \varphi) \quad = - \int_\Omega \partial^{\alpha - e_m} u (\partial_m \varphi) = \int_\Omega \partial^\alpha u \varphi$$

for all $\varphi \in C^\infty(\overline{\Omega})$. Thus $\tilde{u} \in W^{k,p}(\Omega)$.