8.1. Composition of Sobolev functions. Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with C^1 boundary and $1 \leq p < \infty$. Pick $f \in C^1(\mathbb{R})$ with $f' \in L^\infty$. Prove that for $u \in W^{1,p}(\Omega)$, we also have $f \circ u \in W^{1,p}(\Omega)$ and

$$\partial_i(f \circ u) = f'(u)\partial_i u.$$

Hint: Approximate u by smooth functions.

Solution: Denote by $c := \sup_{\mathbb{R}} |f'|$. Then $|f(x) - f(y)| \leq c\,|x - y|$. This implies

$$\|f \circ u - f \circ v\|_{L^p(\Omega)} \leq c\,\|u - v\|_{L^p(\Omega)}\,.$$

for all $u, v \in L^p(\Omega)$. Therefore, if $u_j \in C^\infty(\overline{\Omega})$ converges to $u \in W^{1,p}(\Omega)$ in $W^{1,p}(\Omega)$ and almost everywhere, then $f \circ u_j$ converges to $f \circ u$ in $L^p(\Omega)$. Next consider

$$\|f'(u_j)\partial_i u_j - f'(u)\partial_i u\|_{L^p(\Omega)} \leq \|f'(u_j)(\partial_i u_j - \partial_i u)\|_{L^p(\Omega)} + \|(f'(u) - f'(u_j))\partial_i u\|_{L^p(\Omega)}\,.$$

Then as $\partial_i u_j$ converges to $\partial_i u$ in $L^p(\Omega)$, we have

$$\|f'(u_j)(\partial_i u_j - \partial_i u)\|_{L^p(\Omega)} \leq c\,\|\partial_i u_j - \partial_i u\|_{L^p(\Omega)} \to 0$$

as $j \to \infty$. Furthermore, we have that f' is continuous, whereby $f'(u_j)$ converges almost everywhere to $f'(u)$, as u_j does. In addition, $(f'(u) - f'(u_j))\partial_i u$ is bounded above by $2c\partial_i u \in L^p(\Omega)$, so by dominated convergence, we get

$$\|(f'(u) - f'(u_j))\partial_i u\|_{L^p(\Omega)} \to 0$$

as $j \to \infty$. In conclusion, we have that $f'(u_j)\partial_i u_j$ converges to $f'(u)\partial_i u$ in $L^p(\Omega)$.

As $\partial_i(f \circ u_j) = f'(u_j)\partial_i u_j$, we have that $f \circ u$ is a Cauchy sequence in $W^{1,p}(\Omega)$ which converges in $L^p(\Omega)$ to $f \circ u$ and whose derivatives $\partial_i(f \circ u_j)$ converge to $f'(u)\partial_i u$ in $L^p(\Omega)$. So by uniqueness of limit, $f \circ u \in W^{1,p}(\Omega)$ and $\partial_i(f \circ u) = f'(u)\partial_i u$.

8.2. The absolute value of a Sobolev function. Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with C^1 boundary and $1 \leq p < \infty$. Prove that for $u \in W^{1,p}(\Omega)$, that we also have $|u| \in W^{1,p}(\Omega)$ and that

$$\partial_i\,|u| = \operatorname{sgn}(u)\partial_i u.$$

Hint: Use Exercise 8.1 with $f_\epsilon(x) := \sqrt{x^2 + \epsilon^2} - \epsilon$.

Solution: We have that $0 \leq |x| - f_\epsilon(x) \leq \epsilon$ for $\epsilon > 0$ and $x \in \mathbb{R}$. Also $f_\epsilon'(x) = \frac{x}{\sqrt{x^2+\epsilon^2}}$ converges pointwise to $\operatorname{sgn}(x)$ and $|f_\epsilon'(x)| \leq 1$. So hence $f_\epsilon \in C^1(\mathbb{R})$ and $f_\epsilon' \in L^\infty$.

Therefore by Exercise 8.1, $f_\epsilon \circ u \in W^{1,p}(\Omega)$ and $f_\epsilon \circ u \to |u|$ in $L^p(\Omega)$. Furthermore, $f'_\epsilon(u)\partial_i u$ converges pointwise to $\operatorname{sgn}(u)\partial_i u$ and $f'_\epsilon(u)\partial_i u$ is bounded by $\partial_i u \in L^p(\Omega)$. Thus, by dominated convergence, we have

$$\|f'_\epsilon(u)\partial_i u - \operatorname{sgn}(u)\partial_i u\|_{L^p(\Omega)} \to 0$$

for $\epsilon \to 0$. Hence, by the same argument as at the end of Exercise 8.1, $|u| \in W^{1,p}(\Omega)$ and $\partial_i |u| = \operatorname{sgn}(u)\partial_i u$.

8.3. Iterated Calderòn–Zygmund. Prove that for all $m, n \in \mathbb{N}$ and $1 < p < \infty$, there is a constant $C > 0$ such that

$$\left\|\partial^{2m} u\right\|_{L^p(\mathbb{R}^n)} \leq C \left\|\Delta^m u\right\|_{L^p(\mathbb{R}^n)}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$.

Solution: The case $m = 1$ is exactly the Calderòn–Zygmund inequality. Now assume we already proved the inequality for $m = k - 1$. Then we have for the multi-index α with $|\alpha| = 2k$, that there are multi-indices β of order $|\beta| = 2k - 2$ and γ with $|\gamma| = 2$ such that $\beta + \gamma = \alpha$, and so

$$\begin{aligned}\|\partial^\alpha u\|_{L^p(\mathbb{R}^n)} &= \left\|\partial^\beta \partial^\gamma u\right\|_{L^p(\mathbb{R}^n)} \leq \left\|\partial^{2k-2}(\partial^\gamma u)\right\|_{L^p(\mathbb{R}^n)} \leq C \left\|\Delta^{k-1}(\partial^\gamma u))\right\|_{L^p(\mathbb{R}^n)} \\ &= C \left\|\partial^\gamma(\Delta^{k-1} u))\right\|_{L^p(\mathbb{R}^n)} \leq C' \left\|\Delta\Delta^{m-1} u\right\|_{L^p(\mathbb{R}^n)}\end{aligned}$$

where we used the induction hypothesis in the first line and the Calderòn–Zygmund inequality in the second line. By summing over all multi-indices of order $2k$, we get the desired result.

8.4. Schwartz space and Fourier transform. [1] The goal of this exercise is to define the Schwartz space and study the Fourier transform on it. We define norms for $k \in n$ on $C^\infty(\mathbb{R}^n)$ by

$$\|u\|_k := \sup_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} |\partial^\alpha u(x)| \, (1 + |x|)^k.$$

and define

$$\mathcal{S}(\mathbb{R}^n) := \{u \in C^\infty(\mathbb{R}^n) : \|u\|_k < \infty \text{ for all } k \in \mathbb{N}\}.$$

[1]This and the next Exercise are simply a lot of checking, don't worry ;)

This is a complete, topological vector space with respect to the distance function

$$d(u,v) := \sum_{k\geq 1} 2^{-k} \frac{\|u-v\|_k}{1+\|u-v\|_k}$$

for $u, v \in \mathcal{S}(\mathbb{R}^n)$.[2]

(a) Prove that $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset \bigcap_{1\leq p\leq\infty} L^p(\mathbb{R}^n)$. We state as a fact that these inclusions are continuous and have dense image in the respective distance functions.

Hint: Use the fact that $\frac{1}{(1+|x|)^{mp}}$ is in $L^1(\mathbb{R}^n)$ whenever $mp > n$.

(b) Prove that for $u, v \in \mathcal{S}(\mathbb{R}^n)$ and for P a polynomial, we get that Pu, uv and $\partial^\alpha u$ are elements of $\mathcal{S}(\mathbb{R}^n)$.

(c) Prove for $u \in \mathcal{S}(\mathbb{R}^n)$ that its Fourier transform $\hat{u} \in \mathcal{S}(\mathbb{R}^n)$. Also prove that if $\lim_{k\to\infty} d(u_k, u) = 0$ for $u_k, u \in \mathcal{S}(\mathbb{R}^n)$, then also $\lim_{k\to\infty} d(\hat{u_k}, \hat{u}) = 0$.

(d) Prove that the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a linear isomorphism of topological vector spaces.

Hint: Use the Fourier inverse formula, which says that $u = \tilde{\mathcal{F}}(\mathcal{F}(u))$ for $u \in \mathcal{S}(\mathbb{R}^n)$ and $\tilde{\mathcal{F}}(u)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \hat{u}(\xi)\ \mathrm{d}\xi$.

Solution:

(a) Let $u \in C_0^\infty(\mathbb{R}^n)$, then the function $x^\alpha \partial^\beta u \in C_0^\infty(\mathbb{R}^n)$, hence is in $L^\infty(\mathbb{R}^n)$. As the sup in $\|\cdot\|_k$ runs over finitely many indices, this proves that $\|u\|_k < \infty$ for all $k \in \mathbb{N}$.

Now let $u \in \mathcal{S}(\mathbb{R}^n)$, then $\|u\|_{L^\infty} \leq \|u\|_1 < \infty$ and so $u \in L^\infty(\mathbb{R}^n)$. Furthermore, for $p < \infty$, we have for $m \in \mathbb{N}$ with $mp > n$, that

$$\int_{\mathbb{R}^n} |u|^p = \int_{\mathbb{R}^n} (|u|\,(1+|x|)^m)^p \frac{1}{(1+|x|)^{mp}} \leq \|u\|_m^p \int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{mp}} < \infty$$

and so $u \in L^p(\mathbb{R}^n)$.

(b) That Pu and $\partial^\alpha u$ are in $\mathcal{S}(\mathbb{R}^n)$, is not difficult to see, as Schwartz functions are C^∞ functions, whose derivatives all go to zero quicker than any polynomial. Let us check for uv. Fix a multi-index β and $k \in \mathbb{N}$ with $|\beta| \leq k$, then

$$(1+|x|)^k \partial^\alpha (uv) = (1+|x|)^k \sum_{\beta+\gamma=\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \partial^\beta u \partial^\gamma u$$

and so

$$\|uv\|_k \leq C \|u\|_k \|v\|_k < \infty.$$

[2]This is an example of a Fréchet space, a way of generalising Banach spaces.

(c) It is a little exercise to see that the expressions $\|u\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^n} \left|x^\alpha \partial^\beta u(x)\right| < \infty$ for all multi-indices α, β if and only if $\|u\|_k < \infty$ for all $k \in \mathbb{N}$.

We have $\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} u(x)\ \mathrm{d}x$ and so for multi-indices β and α, we get

$$\begin{aligned}\xi^\alpha \partial_\xi^\beta \hat{u}(\xi) &= \int_{\mathbb{R}^n} (\xi^\alpha \partial_\xi^\beta e^{-i\langle x,\xi\rangle}) u(x)\ \mathrm{d}x = \int_{\mathbb{R}^n} (\xi^\alpha (-i)^{|\beta|} x^\beta e^{-i\langle x,\xi\rangle}) u(x)\ \mathrm{d}x \\ &= \int_{\mathbb{R}^n} (i^{|\alpha|} \partial_x^\alpha e^{-i\langle x,\xi\rangle})(-i)^{|\beta|} x^\beta u(x)\ \mathrm{d}x \\ &= \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} (-i)^{|\beta|+|\alpha|} \partial_x^\alpha x^\beta u(x)\ \mathrm{d}x = (-i)^{|\beta|+|\alpha|} (\widehat{\partial_x^\alpha x^\beta} u(x))\end{aligned}$$

where in the penultimate line, we used integration by parts, which is justified, as the boundary term will be of the form polynomial times Schwartz function, to goes to zero at ∞. Therefore,

$$\sup_{\xi\in\mathbb{R}^n} \left|\xi^\alpha \partial_\xi^\beta \hat{u}(\xi)\right| \leq \left\|\partial_x^\alpha x^\beta u\right\|_{L^1(\mathbb{R}^n)} < \infty$$

where we used (a) and (b), to conclude that $\partial_x^\alpha x^\beta u \in L^1(\mathbb{R}^n)$. By the same token, if $\lim_{k\to\infty} d(u_k, u) = 0$, then

$$\begin{aligned}\sup_{\xi\in\mathbb{R}^n} \left|\xi^\alpha \partial_\xi^\beta (\hat{u}_k(\xi) - \hat{u}(\xi))\right| &\leq \left\|\partial_x^\alpha x^\beta (u_k - u)\right\|_{L^1(\mathbb{R}^n)} \\ &\leq \sup_{x\in\mathbb{R}^n} \left|(1+|x|)^{n+1} \partial_x^\alpha x^\beta (u_k(x) - u(x))\right| \left\|\frac{1}{(1+|x|)^{n+1}}\right\|_{L^1(\mathbb{R}^n)} \to 0\end{aligned}$$

as $k \to \infty$. Hence also $\lim_{k\to\infty} d(\hat{u_k}, \hat{u}) = 0$.

(d) It is well defined by the previous point. We have by the Fourier inverse formula that $\mathcal{F}(\mathcal{F}u)(x) = (2\pi)^n u(-x)$ for $u \in \mathcal{S}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and so $\mathcal{F}^4 u = (2\pi)^{2n} u$ for $u \in \mathcal{S}(\mathbb{R}^n)$. Therefore, $\mathcal{F}$ is bijective and linear. Again by the previous point, we also know that $\mathcal{F}$ is continuous and its inverse $(2\pi)^{-2n}(\mathcal{F})^3$ is also continuous.

8.5. A generalised Fourier transform.

(a) Prove that for $u, v \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} \hat{u}v = \int_{\mathbb{R}^n} u\hat{v}.$$

By the same token prove

$$\int_{\mathbb{R}^n} u\bar{v} = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{u}\bar{\hat{v}}.$$

This gives you Plancherel's identity

$$\|u\|_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \|\hat{u}\|_{L^2(\mathbb{R}^n)}.$$

(b) Extend $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ in a unique way to an isomorphism $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ with

$$(2\pi)^{-n/2} \|\mathcal{F}(u)\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)} .$$

Prove that this agrees with $\hat{u}$ for $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

Hint: For the second statement, start with u having compact support and mollifiers to get $\hat{u} = \mathcal{F}(u)$ on every compact set, next try to deduce the general case from this special case.

(c) Introduce $\mathcal{S}(\mathbb{R}^n)'$ the space of continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$. As $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, this is a subset of distributions called tempered distributions. They should be thought of as the distributions you can apply Fourier transform to. Prove that

$$T_f : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C} : u \mapsto \int_{\mathbb{R}^n} fu$$

is a tempered distribution for all $f \in L^p(\mathbb{R}^n)$ and $1 \leq p \leq \infty$.

(d) Introduce for $T \in \mathcal{S}(\mathbb{R}^n)'$, its Fourier transform $\mathcal{F}(T)$ by setting

$$\langle \mathcal{F}(T), u \rangle := \langle T, \hat{u} \rangle .$$

Prove that $\mathcal{F}(T) \in \mathcal{S}(\mathbb{R}^n)'$.

(e) Prove that for $f \in L^2(\mathbb{R}^n)$, we have

$$\mathcal{F}(T_f) = T_{\mathcal{F}(f)}.$$

and that for $f \in L^1(\mathbb{R}^n)$, we have

$$\mathcal{F}(T_f) = T_{\hat{f}}.$$

So the Fourier transform on tempered distributions generalises both notions of Fourier transform.

(f) Prove that for $T \in \mathcal{S}(\mathbb{R}^n)'$, the functions

$$x^\alpha T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C} : u \mapsto T(x^\alpha u)$$

and

$$\partial^\beta T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C} : u \mapsto (-1)^{|\beta|} T(\partial^\beta u)$$

are tempered distributions for all multi-indices α, β. Prove that $\mathcal{F}(\partial_j T) = i x_j \mathcal{F}(T)$ and $\mathcal{F}(i x_j T) = -\partial_j \mathcal{F} T$.

Solution:

(a) We have for $u, v \in \mathcal{S}(\mathbb{R}^n)$ that

$$\int_{\mathbb{R}^n} \hat{u}(x)v(x)\ \mathrm{d}x = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}u(\xi)v(x)\ \mathrm{d}\xi\ \mathrm{d}x = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}u(\xi)v(x)\ \mathrm{d}x\ \mathrm{d}\xi$$
$$= \int_{\mathbb{R}^n} u(\xi)\hat{v}(\xi)\ \mathrm{d}\xi.$$

Now set $v = (2\pi)^{-n}\overline{\hat{w}}$ in the previous result, to get

$$(2\pi)^{-n}\int_{\mathbb{R}^n} \hat{u}\overline{\hat{w}} = (2\pi)^{-n}\int_{\mathbb{R}^n} u\hat{\overline{\hat{w}}}$$

and we calculate

$$(2\pi)^{-n}\hat{\overline{\hat{w}}}(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}\overline{\hat{w}}(\xi)\ \mathrm{d}\xi = \overline{(2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\overline{\hat{w}}(\xi)\ \mathrm{d}\xi} = \overline{w}(x),$$

which proves the wanted formula.

(b) Let $u_j \in \mathcal{S}(\mathbb{R}^n)$ be a sequence that converges to $u \in L^2(\mathbb{R}^n)$ in L^2. Then $\mathcal{F}u_j$ is a Cauchy sequence in L^2 by Plancherel. Hence, there is an element $\mathcal{F}u \in L^2(\mathbb{R}^n)$ such that $\|u\|_{L^2(\mathbb{R}^n)} = (2\pi)^{-n}\|\hat{u}\|_{L^2(\mathbb{R}^n)}$ and that $\mathcal{F}u_j$ converges in L^2 to $\mathcal{F}u$. Now let $v_j \in \mathcal{S}(\mathbb{R}^n)$ be another sequence converging to u in L^2. Then $u_j - v_j$ converges to zero in L^2, which means that $\mathcal{F}(u_j - v_j)$ also converges to zero in L^2 by Plancherel. Hence $\mathcal{F}(u)$ is well defined and is an isomorphism as we may also extend the Fourier inverse $\tilde{\mathcal{F}}$ to L^2 in the same way, and then directly see that this is a linear isomorphism where linearity follows by uniqueness of limits. It is continuous, because of the Plancherel identity.

For the prove that $\hat{}$ and $\mathcal{F}$ agree on $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, we first consider u to have compact support. Then for a mollifying kernel η_δ, we get that $u_\delta := u * \eta_\delta \in C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ converges to u both in L^1 and L^2. Hence, we also have that $\hat{u}_\delta = \mathcal{F}(u_\delta) \to \mathcal{F}u$ as $\delta \to 0$. On the other hand, we have that

$$\|\hat{u}_\delta - \hat{u}\|_{L^\infty(\mathbb{R}^n)} \leq \|u_\delta - u\|_{L^1(\mathbb{R}^n)} \to 0$$

as $\delta \to 0$. From $L^\infty(\mathbb{R}^n)$, we get for any compact subset $K \subset \mathbb{R}^n$ that $\hat{u}_\delta$ converges to $\hat{u}$ in $L^2(K)$. Therefore, by uniqueness of limit in $L^2(K)$, we get that $\hat{u} = \mathcal{F}u$ in K. As $\mathbb{R}^n$ is σ-compact, we have $\hat{u} = \mathcal{F}u$. Now if $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then for $u_r\mathbb{1}_{B_r(0)}$, we already know that $\hat{u_r} = \mathcal{F}u_r$. By dominated convergence, u_r converges to u both in L^1 and L^2. As before for $r \to \infty$, we then have that $\hat{u_r}\mathcal{F}(u_r) \to \mathcal{F}u$ in $L^2(\mathbb{R}^n)$ and $\hat{u}_r \to u$ in $L^\infty(\mathbb{R}^n)$. So the same argument with compact sets, we get that $\mathcal{F}u = \hat{u}$.

(c) These being linear continuous, it is enough to check continuity at zero for T_f. Both being metric spaces, it is enough to check continuity at zero using sequences. Let $u_j \in \mathcal{S}(\mathbb{R}^n)$ be a sequence with $\lim_{j\to\infty} \|u_j\|_k = 0$ and see that

$$|T_f(u_j)| = \left|\int_{\mathbb{R}^n} f(x)u_j(x)\ \mathrm{d}x\right| \leq \|f\|_{L^p} \|u_j\|_{L^q}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. If $q = \infty$, then $\|u_j\|_{L^q} \leq \|u_j\|_1 \to 0$. Whereas if $q < \infty$, we have the usual trick with $\|u_j\|_{L^q} \leq \left\|\frac{1}{(1+|x|)^{mq}}\right\|_{L^1(\mathbb{R}^n)}^{1/q} \|u_j\|_m \to 0$ for $mq > n$.

(d) If $\|u_j\|_k \to 0$ for $k \in \mathbb{N}$ as $j \to \infty$, then also $\|\hat{u}_j\|_k \to 0$ for $k \in \mathbb{N}$ as $j \to \infty$. Therefore, as $T \in \mathcal{S}(\mathbb{R}^n)'$, we get

$$\lim_{j\to\infty} |\mathcal{F}(T(u_j))| = \lim_{j\to\infty} |T(\hat{u}_j)| = 0.$$

Therefore, $\mathcal{F}(T)$ being linear is in $\mathcal{S}(\mathbb{R}^n)'$.

(e) Both arguments work in the same way. Let us take for example $f \in L^2(\mathbb{R}^n)$. Then approximate f by function $f_j \in C_0^\infty(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ norm. Then we have for all $u \in \mathcal{S}(\mathbb{R}^n)$, that

$$\mathcal{F}(T_{f_j})(u) = \int_{\mathbb{R}^n} f_j\hat{u} = \int_{\mathbb{R}^n} \hat{f}_j u = T_{\hat{f}_j}u.$$

By Hölder, we have

$$\left|\mathcal{F}(T_{f_j})(u) - \mathcal{F}(T_f)(u)\right| \leq \|f_j - f\|_{L^2(\mathbb{R}^n)} \|u\|_{L^2(\mathbb{R}^n)} \to 0$$

as $j \to \infty$. On the other hand, as $f_j \to f$ in $L^2(\mathbb{R}^n)$, we also have $\hat{f}_j \to \mathcal{F}(f)$ in $L^2(\mathbb{R}^n)$, so we also have

$$\left|T_{\hat{f}_j}u - T_{\mathcal{F}(f)}u\right| \leq \left\|\hat{f}_j - \mathcal{F}(f)\right\|_{L^2(\mathbb{R}^n)} \|u\|_{L^2(\mathbb{R}^n)} \to 0$$

as $j \to \infty$. Hence $\mathcal{F}(T_f) = T_{\mathcal{F}(f)}$. The only difference to $L^1(\mathbb{R}^n)$ is that now $\hat{f}_j \to \hat{f}$ in $L^\infty(\mathbb{R}^n)$, but this only changes the argument very slightly.

(f) The first two checks follow immediately from the fact that for $u_j \in \mathcal{S}(\mathbb{R}^n)$ with $\lim_{j\to\infty} \|u_j\|_k = 0$ for $k \in \mathbb{N}$, we also have

$$\lim_{j\to\infty} \|x^\alpha u_j\|_k = 0 \qquad \text{and} \qquad \lim_{j\to\infty} \left\|\partial^\beta u_j\right\|_k = 0.$$

The other result follows by establishing these rules for $u \in \mathcal{S}(\mathbb{R}^n)$. Indeed, we have for example

$$\begin{aligned}\mathcal{F}(\partial_i u)(\xi) &= \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}(\partial_x^{e_i}u)(x)\ \mathrm{d}x = -\int_{\mathbb{R}^n} (\partial_x^{e_i}e^{-i\langle x,\xi\rangle})u(x)\ \mathrm{d}x \\ &= \int_{\mathbb{R}^n} i\xi_i e^{-i\langle x,\xi\rangle}u(x)\ \mathrm{d}x = i\xi_i\mathcal{F}(u).\end{aligned}$$

Therefore,

$$\mathcal{F}(ix_iT)(u) = (ix_iT)(\hat{u}) = T(i\xi_i\hat{u}) = T(\mathcal{F}(\partial_i u)) = -\partial_i\mathcal{F}(T)(u).$$

Similarly, from $\mathcal{F}(ix_iu)(\xi) = -\partial_i\mathcal{F}(u)$, follows that $\mathcal{F}(\partial_jT) = ix_j\mathcal{F}(T)$.

8.6. Calderòn–Zygmund inequality via multipliers. We reprove the Calderòn–Zygmund inequality by using the Mikhlin multiplier theorem. Prove the following steps. Let $f \in C_0^\infty(\mathbb{R}^n)$.

(a) Prove that the function $K_j * f$ defines a tempered distribution in the usual way. Call this distribution T_j.

(b) Prove that $-\sum_{i=1}^n (x_i^2\mathcal{F}(T_j)) = i\xi_iT_{\hat{f}}$, by using the identity $\Delta(K_j * f) = \partial_j f$.

(c) Prove that $\mathcal{F}(\partial_i(K_j * f)) = m_{ij}\hat{f}$ where $m_{ij}(\xi) := \frac{\xi_i\xi_j}{|\xi|^2}$.

(d) Deduce the Calderòn–Zygmund inequality from Mikhlin multiplier theorem.

Solution:

(a) Recall $K_j(x) = \frac{x_j}{|x|^n\omega_n}$. For $f \in C_0^\infty(\mathbb{R}^n)$, we have that

$$|K_j * f|(x) \leq \int_{\mathbb{R}^n} \left|\frac{x_j - y_j}{|x-y|^n\omega_n}\right| |f(y)| \, \mathrm{d}y \leq C\frac{1}{(1+|x|)^{n-1}}.$$

So $K_j * f$ goes to zero as the radius goes to ∞, so $K_j * f \in L^\infty(\mathbb{R}^n)$, which defines a tempered distribution $T_j = T_{K_j*f}$. Notice, that this estimate is not enough to prove $K_j * f$ is $L^2(\mathbb{R}^n)$ if $n = 2$.

(b) We have

$$\Delta(T_j) = T_{\Delta(K_j*f)} = T_{\partial_j f} = \partial_j(T_f),$$

therefore by applying the Fourier transform to these tempered distributions, we get by the rules in (f) of the previous exercise,

$$\sum_{i=1}^n((ix_i)^2\mathcal{F}(T_j)) = ix_j\mathcal{F}(T_f).$$

More in formulae, we have for $\psi \in \mathcal{S}(\mathbb{R}^n)$ that if we take $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\Delta\hat{\varphi} = \hat{\psi}$, or $-|\xi|^2\varphi(\xi) = \psi(\xi)$ for $\xi \in \mathbb{R}^n$, that

$$\begin{aligned}\int_{\mathbb{R}^n}(K_j * f)\hat{\psi} &= \int_{\mathbb{R}^n}(K_j * f)\Delta\hat{\varphi} = \int_{\mathbb{R}^n}\Delta(K_j * f)\hat{\varphi} \\ &= \int_{\mathbb{R}^n}\partial_j f\hat{\varphi} = \int_{\mathbb{R}^n} i\xi_j\hat{f}\varphi = \int_{\mathbb{R}^n} i\xi_j\hat{f}(-|\xi|^{-2})\psi\end{aligned}$$

Notice that the notation $\mathcal{F}(K_j * f)(\xi) = -i\xi_i \left|\xi\right|^{-2} \hat{f}$ is very sloppy and makes actually only sense in the distributional sense, as we can easily put a function $\hat{f} \in L^2(\mathbb{R}^n)$ such that $-i\xi_i \left|\xi\right|^{-2} \hat{f} \notin L^2(\mathbb{R}^n)$.

(c) Now we want to calculate the Fourier transform of a function $\partial_i(K_j * f)$ which we know to lie in $L^2(\mathbb{R}^n)$ by the lecture course. So as the inclusion of $L^1_{loc}(\mathbb{R}^n)$ is still injective and as we have proven the different Fourier transform to agree, we can also simply calculate the Fourier transform of $\mathcal{F}(T_{\partial_i(K_j * f)})$. Thus we get

$$-\sum_{k=1}^{n}((x_k)^2\mathcal{F}(T_{\partial_i(K_j*f)})) = -\sum_{k=1}^{n}((x_k)^2 ix_i\mathcal{F}(T_j)) = ix_j(ix_j\mathcal{F}(T_f)).$$

As all the functions involved are L^2 functions, we get the equation

$$\mathcal{F}(\partial_i(K_j * f)) = m_{ij}\mathcal{F}(f).$$

(d) m_{ij} verifies the condition of the multiplier theorem, so we have that for $1 < p < \infty$, there is a constant $C := C(n,p) > 0$ such that for every $u \in C_0^\infty(\mathbb{R}^n)$, we have

$$\left\|\partial_i(K_j * f)\right\|_{L^p} \leq C \left\|f\right\|_{L^p}.$$

We also know that $\partial_i(K_j * \Delta u) = \partial_i\partial_j u$ for every $u \in C_0^\infty(\mathbb{R}^n)$ and so plugging in $f := \Delta u$ into the previous inequality gives exactly the Calderòn–Zygmund inequalities.