9.1. Calderòn–Zygmund fails for $p = 1$. Let $\rho : \mathbb{R}^2 \to \mathbb{R}$ be a smooth cut-off function, equal to 1 on the unit disc $B_1(0)$ with compact support in $B_2(0)$ with values in $[0, 1]$. For $0 < \epsilon < 1$ define $u_\epsilon : \mathbb{R}^2 \to \mathbb{R}$ by $u_\epsilon(x, y) := \rho(x, y) \log(x^2 + y^2 + \epsilon^2)$. Prove that

$$\sup_{0<\epsilon<1} \|\Delta u_\epsilon\| < \infty, \qquad \lim_{\epsilon \to 0} \|\partial_x \partial_y u\|_{L^1} = \infty.$$

9.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $1 < p < \infty$. Prove that there is a constant $C > 0$ such that for all $u, f, f_1, \ldots, f_n \in C_0^\infty(\Omega)$ with $\Delta u = f + \sum_{i=1}^n \partial_i f_i$, we have

$$\|\nabla u\|_{L^p} \le C \left(\|f\|_{L^p} + \sum_{i=1}^n \|f_i\|_{L^p} \right).$$

Prove the same estimate with Δ replaced by any homogeneous elliptic operator with constant coefficients $Lu = \sum_{j,i=1}^n a_{ij} \partial_{ij}^2 u$.

Hint: For the operator L, use $x \to u(Bx)$ for B a square matrix to reduce it to the case of Δ.

9.3. Dual of Sobolev spaces Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. We define

$$W^{-1,p}(\Omega) := (W_0^{1,q}(\Omega))^*.$$

Now define for $f \in L^p(\Omega)$, $\Phi_f \in W^{-1,p}(\Omega)$ by

$$\Phi_f(v) := \int_\Omega fv$$

for $v \in W_0^{1,q}(\Omega)$. Prove that the map $\kappa : L^p(\Omega) \to W^{-1,p}(\Omega) : f \to \Phi_f$ is the dual operator to the inclusion $\iota : W_0^{1,q}(\Omega) \hookrightarrow L^q(\Omega)$. Deduce that it is a compact injective operator with dense image.

9.4. Review older exercises Last week's exercise sheet was admittedly a bit long, so if you did not have time to finish it during last week, you can go back to it now ;) Or simply relax and enjoy the sun :)

Please hand in your solutions for this sheet by Monday 02/05/2016.