9.1. Calderòn–Zygmund fails for $p = 1$. Let $\rho : \mathbb{R}^2 \to \mathbb{R}$ be a smooth cut-off function, equal to 1 on the unit disc $B_1(0)$ with compact support in $B_2(0)$ with values in $[0, 1]$. For $0 < \epsilon < 1$ define $u_\epsilon : \mathbb{R}^2 \to \mathbb{R}$ by $u_\epsilon(x, y) := \rho(x, y) \log(x^2 + y^2 + \epsilon^2)$. Prove that

$$\sup_{0<\epsilon<1} \|\Delta u_\epsilon\| < \infty, \qquad \lim_{\epsilon\to 0} \|\partial_x \partial_y u\|_{L^1} = \infty.$$

Solution: We calculate

$$\nabla u_\epsilon(x, y) = \nabla\rho(x, y)u_\epsilon(x, y) + \frac{\rho(x, y)}{x^2 + y^2 + \epsilon^2} \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

and

$$\begin{aligned}
\partial_x^2 u_\epsilon(x, y) &= \partial_x^2\rho(x, y)u_\epsilon(x, y) + 2\partial_x\rho(x, y)\frac{2x}{x^2 + y^2 + \epsilon^2} + \frac{\rho(x, y)2(-x^2 + y^2 + \epsilon^2)}{(x^2 + y^2 + \epsilon^2)^2} \\
\partial_y^2 u_\epsilon(x, y) &= \partial_y^2\rho(x, y)u_\epsilon(x, y) + 2\partial_y\rho(x, y)\frac{2y}{x^2 + y^2 + \epsilon^2} + \frac{\rho(x, y)2(x^2 - y^2 + \epsilon^2)}{(x^2 + y^2 + \epsilon^2)^2} \\
\Delta u_\epsilon(x, y) &= \Delta\rho(x, y)u_\epsilon(x, y) + \frac{4(\partial_x\rho(x, y)x + \partial_y\rho(x, y)y)}{x^2 + y^2 + \epsilon^2} + \rho(x, y)\frac{4\epsilon^2}{(x^2 + y^2 + \epsilon^2)^2}
\end{aligned}$$

We now look at $\|\Delta u_\epsilon\|_{L^1}$. We see easily that the part on $\mathbb{R}^2 \setminus B_1(0)$ is bounded uniformly by some constant independent of ϵ. For the part over $B_1(0)$, we have

$$\int_{B_1(0)} |\Delta u_\epsilon| = 2\pi \int_0^1 \frac{4\epsilon^2 r}{(r^2 + \epsilon^2)^2}\, \mathrm{d}r = 2\pi \left[\frac{-2\epsilon^2}{r^2 + \epsilon^2}\right]_0^1 = 2\pi[\frac{-2\epsilon^2}{\epsilon^2 + 1} + 2] \leq 2.$$

So $\sup_{0<\epsilon<1} \|\Delta u_\epsilon\| < \infty$. On the other hand,

$$\begin{aligned}
\|\partial_x\partial_y u_\epsilon\|_{L^1} &\geq \int_{B_1(0)} \frac{4\,|xy|}{(x^2 + x^2 + \epsilon^2)^2} = \int_0^1 \int_0^{2\pi} \frac{2\,|\sin(2\theta)|\, r^3}{(r^2 + \epsilon^2)^2}\, \mathrm{d}\theta \mathrm{d}r \\
&= 8 \int_0^1 \frac{r^3}{(r^2 + \epsilon^2)^2}\, \mathrm{d}r = 8 \left[\frac{\epsilon^2}{2(\epsilon^2 + r^2)} + \frac{1}{2}\log(\epsilon^2 + r^2)\right]_0^1 \to \infty
\end{aligned}$$

as $\epsilon \to 0$. So $\lim_{\epsilon\to 0} \|\partial_x\partial_y u\|_{L^1} = \infty$.

9.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $1 < p < \infty$. Prove that there is a constant $C > 0$ such that for all $u, f, f_1, \ldots, f_n \in C_0^\infty(\Omega)$ with $\Delta u = f + \sum_{i=1}^n \partial_i f_i$, we have

$$\|\nabla u\|_{L^p} \leq C \left(\|f\|_{L^p} + \sum_{i=1}^n \|f_i\|_{L^p}\right).$$

Prove the same estimate with Δ replaced by any homogeneous elliptic operator with constant coefficients $Lu = \sum_{j,i=1}^n a_{ij}\partial^2_{ij}u$.

Hint: For the operator L, use $x \to u(Bx)$ for B a square matrix to reduce it to the case of Δ.

Solution: For Δ, we start by writing the relation

$$u = K * f + \sum_{i=1}^n (K_i * f_i)$$

So we have

$$\partial_j u = K_j * f + \sum_{i=1}^n \partial_j (K_i * f_i).$$

Thus

$$\|\partial_j u\|_{L^p} \leq \|\partial_j (K * f)\|_{L^p} + \sum_{i=1}^n \|\partial_j (K_i * f_i)\|_{L^p}$$

By Calderón–Zygmund, we know that

$$\|\partial_j (K_i * f_i)\|_{L^p} \leq C \|f_i\|_{L^p}.$$

For the other term, we use the Gagliardo–Nirenberg interpolation theorem

$$\begin{aligned}\|\partial_j (K * f)\|_{L^p} &\leq C_1 \|K * f\|_{L^p}^{1/2} \left\|\partial^2 (K * f)\right\|_{L^p}^{1/2} \\ &\leq C_1 (\|K * f\|_{L^p} + \left\|\partial^2 (K * f)\right\|_{L^p}) \\ &\leq C_1 (\|K\|_{L^1(\Omega)} \|f\|_{L^p} + C_2 \|f\|_{L^p}) \leq C_3 \|f\|_{L^p}\end{aligned}$$

where in the last line we used Young's inequality and Calderòn–Zygmund inequality. Summing all these inequalities, this gives the wanted inequality.

For $L = \sum_{j,i=1}^n a_{ij}\partial^2_{ij}$, we have that for $v(x) = u(Bx)$ for B an invertible square matrix, we have

$$\mathrm{Hess}(v)(x) = B^\top \mathrm{Hess}(u)(Bx) B$$

Hence,

$$\Delta v(x) = \mathrm{tr}(B^\top \mathrm{Hess}(u)(Bx) B) = \mathrm{tr}(\mathrm{Hess}(u)(Bx) B B^\top)$$

Hence, if we choose $B^2 = A$, we get $\Delta v(x) = (Lu)(Bx)$. Thus, if $Lu = f + \sum_{i=1}^n \partial_i f_i$, then for $v(x) = u(Bx), g(x) = f(Bx), g_j(x) = \sum_{i=1}^n (B^{-1})_{ji} f_i(Bx))$, we have

$$\Delta v(x) = Lu(Bx) = f(Bx) + \sum_{i=1}^n (\partial_i f_i)(Bx) = g(x) + \sum_{j=1}^n \partial_j g_j(x).$$

Therefore, since B is non-degenerate, and by $\|u \circ B\|_{L^p} \det B = \|u\|_{L^p}$, we now get

$$\|\nabla u\|_{L^p} \le C_1 \|\nabla v\|_{L^p} \le C_2 \left(\|g\|_{L^p} + \sum_{i=1}^n \|g_i\|_{L^p} \right) \le C_3 \left(\|f\|_{L^p} + \sum_{i=1}^n \|f_i\|_{L^p} \right).$$

9.3. Dual of Sobolev spaces Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. We define

$$W^{-1,p}(\Omega) := (W_0^{1,q}(\Omega))^*.$$

Now define for $f \in L^p(\Omega)$, $\Phi_f \in W^{-1,p}(\Omega)$ by

$$\Phi_f(v) := \int_\Omega fv$$

for $v \in W_0^{1,q}(\Omega)$. Prove that the map $\kappa : L^p(\Omega) \to W^{-1,p}(\Omega) : f \to \Phi_f$ is the dual operator to the inclusion $\iota : W_0^{1,q}(\Omega) \hookrightarrow L^q(\Omega)$. Deduce that it is a compact injective operator with dense image.

Solution: We start by proving that κ is well defined and bounded. Indeed, let $f \in L^p(\Omega)$ and $v \in W_0^{1,q}(\Omega)$, then

$$\left| \int_\Omega fv \right| \le \|f\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)} \le C \|f\|_{L^p(\Omega)} \|v\|_{W_0^{1,q}(\Omega)},$$

where we used Poincaré inequality.

Then as

Now let us prove that κ is the dual of ι. We recall that

$$L^p(\Omega) \to (L^q(\Omega))^* : f \to (g \mapsto \int_\Omega fg)$$

is an isomorphism. Under this identification, we can now write

$$\langle \iota^* f, v \rangle := \langle f, \iota v \rangle = \int_\Omega fv = \Phi_f(v) = \langle \kappa f, v \rangle$$

for all $v \in W_0^{1,q}(\Omega)$ and $f \in L^p(\Omega)$. Hence $\iota^* = \kappa$.

Now we can deduce the wanted properties of κ by using the related properties of ι by using some theorems from Functional Analysis I.

- injective: As the image of ι contains $C_0^\infty(\Omega)$, ι has dense image and so by 4.1.8, κ is injective.
- dense image: As $L^q(\Omega)$ and $W_0^{1,q}(\Omega)$ are reflexive, we have the natural identification $\kappa^* = \iota$ by 4.1.3. As ι is injective, by 4.1.8, κ has dense image.
- compact: By Rellich's theorem for $p < n$, the inclusion ι is the composition of a compact operator $W_0^{1,q}(\Omega) \to L^{\frac{np}{n-p}}(\Omega)$ with the bounded operator $L^{\frac{np}{n-p}}(\Omega) \to L^p(\Omega)$ as $\frac{np}{n-p} \geq p$, so ι is compact. Similar reasonings give ι is also compact for $p \geq n$. Thus by 4.2.10, κ is also compact.

9.4. Review older Exercises Last week's exercise sheet was admittedly a bit long, so if you did not have time to finish it during last week, you can go back to it now ;) Or simply relax and enjoy the sun :)

Solution: Trivial is the worst word in the vocabulary of a mathematician ;)