Graph Theory

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Assignment 12

To be completed by May 26

Unless noted otherwise, all graphs considered are simple. The solution of every problem should be no longer than one page.

Problem 1: *The point of this exercise is to give an alternative (although less rigorous) proof of Kuratowski's theorem using the theorem of Tutte.*

1. Use stereographic projections (see Chapter 6) to show the following. If e is an edge of the planar graph G then G has a planar drawing with e on the outer face.

2. Prove Kuratowski's theorem using the following ideas:

   - Use Tutte's theorem when G is 3-connected.
   - Apply induction to the graphs G_1, G_2 as defined in the lectures, and use them to find a planar embedding of G
   - When G is 2-connected, contract G_2 to the edge uv to see that $G_1 + uv$ is planar.

Problem 2: The lower bound for $R(p,p)$ that you learn in the lectures is not a constructive proof: it merely shows the *existence* of a red-blue coloring not containing any monochromatic copy of K_p by bounding the number of bad graphs.

Give an explicit coloring on $K_{(p-1)^2}$ that proves $R(p,p) > (p-1)^2$.

Problem 3: Prove that for every fixed positive integer r, there is an n such that any coloring of all the subsets of $[n]$ using r colors contains two non-empty disjoint sets X and Y such that X, Y and $X \cup Y$ have the same color.

Problem 4: Prove that for every $k \geq 2$ there exists an integer N such that every coloring of $[N]$ with k colors contains three distinct numbers a, b, c satisfying $ab = c$ that have the same color.