Graph Theory

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Assignment 13

To be completed by June 2

Unless noted otherwise, all graphs considered are simple. The solution of every problem should be no longer than one page.

Problem 1: Prove the following strengthening of Schur's theorem: for every $k \geq 2$ there is an N such that any k-coloring of $[N]$ contains three *distinct* integers a, b, c of the same color satisfying $a + b = c$.

Problem 2: A transitive tournament is one that does not contain any directed cycle, or equivalently: there is an ordering $v_1, \ldots, v_n$ of the vertices such that the edge $v_i v_j$ is directed from i to j for any $1 \leq i < j \leq n$.

(a) Prove that any tournament on n vertices contains a transitive subtournament on $\lceil \log_2(n) \rceil$ vertices.

(b) Prove, using the probabilistic method, that there is a tournament containing no transitive subtournament on $\lceil 2\log_2(n) \rceil + 2$ vertices.

Problem 3: Let $k, l \geq 1$ be integers and show that any sequence of $kl + 1$ distinct numbers $a_1, \ldots, a_{kl+1}$ contains a monotone increasing subsequence of length $k + 1$ or a monotone decreasing subsequence of length $l + 1$.

[Hint: .rebmun hcae ta gnitrats ecneuqes gnisaerced tsegnol eht ta kooL .$a_i < a_j$ neht $i < j$ secidni rof emas eht si htgnel siht fI]