Graph Theory

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Assignment 2

To be completed by March 10

Unless noted otherwise, all graphs considered are simple. The solution of every problem should be no longer than one page.

Problem 1: Show that in a tree containing an even number of edges, there is at least one vertex with even degree.

Problem 2: Given a graph G and a vertex $v \in V(G)$, $G - v$ denotes the subgraph of G induced by the vertex set $V(G) \setminus \{v\}$. Show that every connected graph G of order at least two contains vertices x and y such that both $G - x$ and $G - y$ are connected.

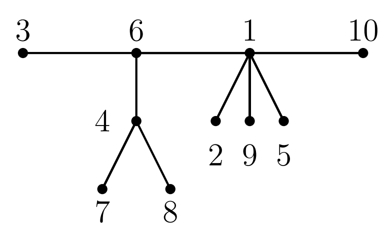
Problem 3: Let T be a tree with $2k$ odd-degree vertices. Prove that T decomposes into k paths (i.e. its edge-set is the disjoint union of k paths).

Problem 4: Prove that a connected graph G is a tree if and only if any family of pairwise (vertex-)intersecting paths $P_1, \ldots, P_k$ in G have a common vertex.

[Hint (backwards): .shtap eerht rof evorp tsriF]

Exercise:

(a) What is the Prüfer code of the following tree? What is the map associated with it in Joyal's proof with left end 4 and right end 5?



(b) Which labeled tree has Prüfer code (5,1,1,7,7,5)?

Problem 5:

(a) Describe which Prüfer codes correspond to stars (i.e. to trees isomorphic to $K_{1,n-1}$).

(b) Describe what trees correspond to Prüfer codes containing exactly 2 values.

Problem 6: Let T be a forest on vertex set $[n]$ with components $T_1, \ldots, T_r$. Prove, by induction on r, that the number of spanning trees on $[n]$ containing T is $n^{r-2} \prod_{i=1}^{r} |T_i|$. Deduce Cayley's formula.

[Hint: T_j dna T_i neewteb egde na niatnoc seert hcus ynam woh tnuoC]