Graph Theory

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Assignment 4

To be completed by March 24

Unless noted otherwise, all graphs considered are simple. The solution of every problem should be no longer than one page.

Problem 1: Show that if $k > 0$ then the edge set of any connected graph with $2k$ vertices of odd degree can be split into k trails.

Problem 2: Let G be a connected graph that has an Euler tour. Prove or disprove the following statements.

(a) If G is bipartite then it has an even number of edges.

(b) If G has an even number of vertices then it has an even number of edges.

(c) For edges e and f sharing a vertex, G has an Euler tour in which e and f appear consecutively.

Problem 3: Let G be a connected graph on n vertices with minimum degree δ. Show that

(a) if $\delta \leq \frac{n-1}{2}$ then G contains a path of length 2δ, and

(b) if $\delta \geq \frac{n-1}{2}$ then G contains a Hamiltonian path.

Problem 4: Let $q > 1$ be an integer and $D = \{d > 1 : d|q\}$ be the set of divisors of q other than 1. Use the previous problem to prove that there is an ordering $d_1, \ldots, d_k$ of the elements of D such that adjacent pairs are not coprime (i.e., $\gcd(d_i, d_{i+1}) > 1$ for every i).

Problem 5: Show that the maximum number of edges in a non-Hamiltonian graph on $n \geq 3$ vertices is $\binom{n-1}{2} + 1$.