Graph Theory

Solutions 1

The aim of the homework problems is to help you understand the theory better by actively using it to solve exercises. ***Do not read the solutions*** *before you believe you have solved the problems: it ruins your best way of preparing for the exam. The purpose of this write-up is merely to provide some guideline on how solutions should look like, and to help clean up hazy arguments. For hints, feel free to consult your teaching assistant.*

Problem 1(c): Let us call the vertices a, b, c, d, e, f, g, h and suppose they are ordered by their degrees (so $d(a) = 1$, $d(b) = 2$ and so on). We have 8 vertices, so each one of degree 6 is connected to all but one of the other vertices. So each of e, f, g and h sends at least one edge to the set $\{a, b\}$, but that is impossible, since a and b are adjacent to at most 3 edges in total. Hence there is no such graph.

Problem 5: We do induction on n. If $n = 7$ then we have $35 - 14 = 21$ edges. The only graph on 7 vertices with 21 edges is K_7, so we can take K_7 as a subgraph satisfying the conditions.

Now suppose that $n > 7$ and we know the statement for $n - 1$. If all the degrees in the graph are at least 6, we can again take the graph itself is a good subgraph. If not, then there is a vertex v with $d(v) \leq 5$. Then $G - v$ is a graph on $n - 1$ vertices, and it has at least $5n - 14 - 5 = 5(n - 1) - 14$ edges (we lose at most 5 by deleting v). So by induction $G - v$ has a good subgraph, but that is also a good subgraph of G.

Problem 6: Suppose there are two vertex-disjoint paths P_1 and P_2 of maximum length l. Since the graph is connected, there is a path Q between P_1 and P_2, say from $w_1 \in P_1$ to $w_2 \in P_2$ such that the interior vertices of Q avoid P_1 and P_2 (or formally: $V(Q) \cap (V(P_1) \cup V(P_2)) = \{w_1, w_2\}$). Here w_1 and w_2 cut P_1 and P_2 into two pieces each. Let P_1' be the longer piece of P_1 and P_2' be the longer piece of P_2, so both P_1' and P_2' have length at least $l/2$. Moreover, $P_1' \cup Q \cup P_2'$ forms a path(!) of length at least $l + e(Q)$, which is impossible because the longest path had length l. This is a contradiction.

To say that $P_1' \cup Q \cup P_2'$ is a path, we really needed the fact that Q is internally disjoint from P_1 and P_2. But why is there such a Q? To show this, take arbitrary vertices $v_1 \in P_1$ and $v_2 \in P_2$. As G is connected, there is a path Q_0 from v_1 to v_2. Since P_1 and P_2 are vertex-disjoint, there is a *last* vertex in Q_0 from P_1, let us call this w_1. We define w_2 to be the first vertex from

P_2 appearing after w_1 in Q_0. Then the part of the path Q_0 between w_1 and w_2 is a good choice for Q.