Graph Theory

Solutions 10

The purpose of this write-up is merely to provide some guideline on how solutions should look like, and to help clean up hazy arguments. For hints for other problems, feel free to consult your teaching assistant.

Problem 3:

Suppose for contradiction that there is a graph G and edge set F that does not satisfy the statement. Add edges one-by-one to G as long as it remains a counterexample, and let us call this new graph G from now on. Note that adding edges will not destroy our minimum degree condition, and this process stops before reaching the complete graph because in K_n every permutation of the vertices gives a Hamilton cycle: it is easy to find one that crosses all the edges in F.

Now we know that G has no Hamilton cycle through F, but if we add some missing edge uv, then the new graph will have one. So G contains some Hamilton path $v_1 \dots v_n$ crossing all the edges of F with $u = v_1$ and $v = v_n$. We know that v_1 has at least $l = \lceil \frac{n+q}{2} \rceil$ neighbors $v_{i_1}, \dots, v_{i_l}$. Similarly, v_n has at least l neighbors among $v_1, \dots, v_{n-1}$, and since $2l - (n-1) \geq q+1$, it must have at least $q+1$ of the $v_{i_j - 1}$ in its neighborhood. This means that there is some j such that $v_{i_j-1}v_{i_j}$ is not in F, but $v_{i_j-1}v_n$ is an edge (and $v_1 v_{i_j}$ is an edge as well). So we get a Hamilton cycle containing all of F by rotating the path: delete the edge $v_{i_j-1}v_{i_j}$ and add $v_1 v_{i_j}$ and $v_{i_j-1}v_n$. This contradicts the definition of G, hence no counterexample to the statement exists.