Graph Theory

Solutions 13

The purpose of this write-up is merely to provide some guideline on how solutions should look like, and to help clean up hazy arguments. For hints for other problems, feel free to consult your teaching assistant.

Problem 1:

We will use an averaging argument to solve the problem. Let G be an extremal graph on $[n]$ (i.e. G has $\mathrm{ex}(n, H)$ edges). Define $G_i = G - i$ for any vertex $i \in [n]$ and m_i to be the number of edges in G_i.

Of course, G_i is H-free, so $m_i \leq \mathrm{ex}(n-1, H)$. On the other hand, any edge of G is in exactly $n-2$ of the G_i (all but the two obtained by deleting an endpoint of the edge). This means that

$$(n-2) \cdot \mathrm{ex}(n, H) = \sum_{i \in [n]} m_i \leq n \cdot \mathrm{ex}(n-1, H).$$

Dividing both sides by $n(n-1)(n-2)/2$, we obtain $\frac{\mathrm{ex}(n,H)}{\binom{n}{2}} \leq \frac{\mathrm{ex}(n-1,H)}{\binom{n-1}{2}}$, what we wanted to show.

Problem 2 (sketch):

We go along the lines of the first proof of Turán's theorem. With G, v, S, T as in the proof, we define the graph H such that $H[S]$ is a Turán graph, $H[T]$ is empty and H contains all $S-T$ edges. We need to show H contains at least as many triangles as G

For triangles in S this follows by induction. For triangles touching some vertex in T this follows from the fact that in a K_{r+1}-free graph, any vertex w is contained in at most $\mathrm{ex}(d(w), K_{r+1})$ triangles (this is an easy application of Turán's theorem, noting that such triangles correspond to edges induced by the neighborhood of w).

This shows that *some* r-partite graph is optimal, and an easy calculation (like in the original setting) shows that the balanced r-partite graph is the best among r-partite graphs.

Problem 3:

Observe, first, that among any 4 points in the plane some three form a non-acute triangle (with an angle of at least 90 degrees). Indeed, if the points are in convex position, then one of the four angles of the quadrilateral is at least 90 degrees (their sum is 360), whereas if their convex hull is a triangle, then one (even two, in fact) of the three angles at the point in the middle is at least 90 degrees (again, their sum is 360).

Now look at the set X, and draw an edge between two points if their distance is greater than $1/\sqrt{2}$. We claim that this graph contains no K_4. Suppose it did, and look at the four points inducing a K_4. By the above claim, three of them form a non-acute triangle. Let us denote the sides by a, b, c, where the angle between a and b is at least 90 degrees. Both a and b have length greater than $1/\sqrt{2}$, so c must have length greater than 1, contradiction.

So this graph we defined is K_4-free, hence, by Turán's theorem, it contains at most $n^2/3$ edges, what we wanted to show.

Problem 4:

Take 8 vertices corresponding to the batteries, and mark a test of two batteries by drawing an edge between them. If there is an edge induced by the four good batteries then there was a successful trial. To make sure this is the case, we want a graph that contains no independent set of size 4.

So our question really is: what is the minimum number of edges in an 8-vertex graph with independence number less than 4, or, equivalently, what is the maximum number of edges in a K_4-free graph on 8 vertices. By Turán's theorem, we know that the answer to the latter question is 21, and so the answer to our original question is 7.