Graph Theory

Solutions 2

The aim of the homework problems is to help you understand the theory better by actively using it to solve exercises. ***Do not read the solutions*** *before you believe you have solved the problems: it ruins your best way of preparing for the exam. The purpose of this write-up is merely to provide some guideline on how solutions should look like, and to help clean up hazy arguments. For hints, feel free to consult your teaching assistant.*

Problem 2: Take a spanning tree T of G. It has at least two leaves, say x and y. Then $T - x$ and $T - y$ are both connected, hence so are their supergraphs, $G - x$ and $G - y$.

Problem 4: Here we only show that if G is a tree then any pairwise intersecting paths $P_1, \ldots, P_k$ intersect at a common vertex. This is trivial for $k \leq 2$, so let us start by showing it for $k = 3$. Suppose P_1, P_2 and P_3 are pairwise intersecting but they have no common vertex. Look at $P_1 = v_1 v_2 \cdots v_l$ and color a vertex v_i red if it is also in P_2, color it blue if it is also in P_3 and leave it uncolored if it is only in P_1. By our conditions, no vertex is colored both red and blue, but there *is* some red vertex v_i and some blue vertex v_j.

We may assume that $i < j$. Then there is some vertex $v_{i'}$ with $i \leq i' < j$ such that $v_{i'}$ is red but $v_{i'+1}$ is not. Then the edge $e = v_{i'} v_{i'+1}$ is neither in P_2, nor in P_3. Now delete e from the tree, it splits the graph into two components. Each of P_2 and P_3 can only be in one of the components, because they do not contain e. But $v_i \in P_2$ and $v_j \in P_3$ are in different components, hence so are P_2 and P_3. Thus they cannot intersect, contradiction.

We could use the same idea to prove the statement for $k > 3$, but let us follow the standard way of proving Helly-type theorems, and use induction instead.

So suppose the statement is true for $k - 1$. Then $P_2, \ldots, P_k$ share a vertex: u, and $P_1, P_3, \ldots, P_k$ also share a vertex: v. But then the unique u-v path in the tree – let us call it P' – is contained in all of $P_3, \ldots, P_k$. So if we can find a vertex shared by P_1, P_2 and P', it will be a good choice for all the paths. Now we can apply the $k = 3$ case to these three paths (the condition clearly holds) to find this common vertex.

Remark. There are many other ways to prove this result. Induction on the number of vertices gives a somewhat easier proof than the one above. Alternatively, one can also use the fact that the intersection of two paths is a path (but a rigorous proof of this and its application to the problem needs care).

Problem 6: If $r = 1$ then T is a tree and there is exactly $1 = n^{1-2}|T|$ spanning tree containing T. Let us do induction on r and assume we know the statement for $r - 1$.

So T has r components, the question is how many ways there are to add the remaining $r - 1$ edges to make it a spanning tree. Well, let us count how many such extensions use some particular edge e between T_i and T_j. If we add e to T, then we get a forest with $r-1$ components whose sizes are $|T_1|, \ldots, |T_{i-1}|, |T_{i+1}|, \ldots, |T_{j-1}|, |T_{j+1}|, \ldots |T_r|$ and $|T_i| + |T_j|$. Then induction tells us that the number of trees extending this is exactly $n^{r-3} \frac{|T_i|+|T_j|}{|T_i||T_j|} \prod_{k=1}^r |T_k|$. There are $|T_i||T_j|$ choices for e (this is the number of missing edges between T_i and T_j), so the number of trees extending T containing an edge between T_i and T_j is exactly $n^{r-3}(|T_i| + |T_j|) \prod_{k=1}^r |T_k|$.

If we sum these up for each pair of components (T_i, T_j), we get

$$\sum_{1 \le i < j \le r} (|T_i| + |T_j|) \cdot n^{r-3} \prod_{k=1}^{r} |T_k| = (r-1) \sum_{i=1}^{r} |T_i| \cdot n^{r-3} \prod_{k=1}^{r} |T_k| = (r-1) n^{r-2} \prod_{k=1}^{r} |T_k|$$

trees, because in the sum $\sum_{1 \le i < j \le r} (x_i + x_j)$, each term x_i is counted $r - 1$ times.

But in the above sum we counted every tree exactly $r - 1$ times. Indeed, as we mentioned, a spanning tree extends T by $r - 1$ edges, and e can be any one of those. In fact, one particular tree was counted once for each choice of the edge e. So to get the actual number of extensions, we need to divide the above formula by $r - 1$. We obtain $n^{r-2} \prod_{k=1}^r |T_k|$, exactly what we wanted to show.