Graph Theory

Solutions 3

The aim of the homework problems is to help you understand the theory better by actively using it to solve exercises. ***Do not read the solutions*** *before you believe you have solved the problems: it ruins your best way of preparing for the exam. The purpose of this write-up is merely to provide some guideline on how solutions should look like, and to help clean up hazy arguments. For hints, feel free to consult your teaching assistant.*

Problem 2: Let X be a set separating u from v. This means that u and v are in different components of $G - X$ – let us call these components C_u and C_v, respectively. Now let us look at a vertex $x \in X$. If x is connected to both C_u and C_v in G, then adding it back to $G - X$ connects the components of u and v, or in other words: $X - x$ is not separating. On the other hand, if x is not connected to both, say it has no neighbor in C_v, then adding it back to $G - X$ cannot join C_u and C_v (in fact, it cannot affect C_v at all). So in this case $X - x$ is still separating.

This means that removing a vertex x from a separating set keeps the set separating iff x had neighbors in both C_u and C_v. Hence X is minimal separating iff all its vertices have neighbors in both C_u and C_v.

Problem 3: Let X be a set of at most $k-1$ vertices. We need to show that $G-X$ is connected. In fact, we are going to show something stronger: $G - X$ has diameter 2. So take two vertices u and v in $G - X$, we want to find a path between them. If they are adjacent vertices then we are done. If not, then note that both of them are adjacent to at least $\frac{n+k-2}{2}$ other vertices in G, and since there are $n - 2$ other vertices in total, these neighborhoods overlap in at least $\frac{n+k-2}{2} + \frac{n+k-2}{2} - (n - 2) = k$ vertices. Now X contains at most $k - 1$ of these k vertices, so there is at least one common neighbor left in $G - X$, providing the path between u and v that we were looking for.

Problem 4: From some Menger-type results in the course, we know that if G is k-connected, v is a vertex and S is a vertex set, then G contains a v-S fan of $min(k, |S|)$ paths.
Exercise: obtain this from Menger's theorem by replacing v with k vertices identical to it

Let us apply this to our 2-connected graph with $k = 2$, $v = y$ and $S = \{x, z\}$, then we obtain internally vertex disjoint paths from y to both x and z. By joining these two paths at

y, we get a path from x to z through y.

For the other direction, suppose G has such a path for all x, y, z, but it is not 2-connected. Then there is some cut vertex x that separates a vertex y from some other vertex z. By the assumption, there is an x-z path P containing y. Now delete the vertex x from G. $P - x$ is still a path in $G - x$. In particular, $G - x$ contains a y-z path (a subpath of $P - x$). But this contradicts our assumption about x separating y from z.