Graph Theory

Solutions 4

The aim of the homework problems is to help you understand the theory better by actively using it to solve exercises. ***Do not read the solutions*** *before you believe you have solved the problems: it ruins your best way of preparing for the exam. The purpose of this write-up is merely to provide some guideline on how solutions should look like, and to help clean up hazy arguments. For hints, feel free to consult your teaching assistant.*

Problem 1: Pair up the odd-degree vertices of the graph and add an edge between any pair. To distinguish them, let us say that the new edges are blue. The resulting *multi*graph is connected with all degrees even, so it has an Euler tour. Now delete the blue edges from this tour: we get k trails that cover each edge of the original graph exactly once.

Alternative solution: We proceed by induction on k. For $k = 1$ the graph has an Euler trail as was shown in class. Now assume the statement is true for $1, \ldots, k-1$ and that the connected graph G has $2k$ odd degrees. Let u and v be two odd-degree vertices and P be a path connecting them. Then if we remove P from G, we get rid of 2 odd degrees. So we want to apply induction on $G - P$, but it might not be connected. Never mind that, we can apply induction on each of the components of $G - P$. Or can we?

Each component will have an even number of odd-degree vertices, say the i'th component has $2k_i$, and here $\sum_{2k_i} = 2k - 2$. By induction we can split the i'th component into k_i trails.. except if $k_i = 0$! But then it has an Euler tour. Note that P must touch each new component, so we can extend P into a trail using this Euler tour. We do this for all such i and we apply induction on each component with $k_i > 0$.

Problem 5: The graph that is the union of K_{n-1} and an edge attaching it to the n'th vertex shows that there is a non-Hamiltonian graph with $\binom{n-1}{2} + 1$ edges. (It's non-Hamiltonian because the degree of the n'th vertex is less than 2.)

Now we need to show that any graph G containing at least $\binom{n-1}{2} + 2$ edges (i.e., missing at most $n - 3$ edges) is Hamiltonian. We use Ore's condition for this. So let u, v be two non-adjacent vertices, we need to show that $d(u) + d(v) \geq n$. K_n contains $2(n-2) = 2n - 4$ edges connecting u and v to the other vertices. But G misses at most $n - 4$ of them (the edge uv is already missing), so contains at least n of these edges. This is exactly what we wanted to show.