Graph Theory

Solutions 5

The aim of the homework problems is to help you understand the theory better by actively using it to solve exercises. ***Do not read the solutions*** *before you believe you have solved the problems: it ruins your best way of preparing for the exam. The purpose of this write-up is merely to provide some guideline on how solutions should look like, and to help clean up hazy arguments. For hints, feel free to consult your teaching assistant.*

Problem 3: If there are such subsets then the inequality clearly holds: $\cup_{i\in I} A_i \subseteq \cup_{i\in I} D_i$ by construction for all $I \subseteq [n]$, and $|\cup_{i\in I} D_i| = \sum_{i\in I} |D_i| = \sum_{i\in I} d_i$ because the D_i are disjoint.

To show the other direction, let us define the following bipartite graph. B will be the disjoint union of the sets B_i of size d_i, and we connect every vertex $v \in B_i$ to all vertices in $A_i \subseteq A$. Our aim is to show that this graph has a matching M covering B. Then taking $D_i = N_M(B_i)$ as the matched image of B_i will work: by construction, $D_i \subseteq A_i$, and since M is a matching, the D_i will be disjoint and will have size $|B_i| = d_i$.

So we just need to check that Hall's condition holds. Take a set $S \subseteq B$. Define $I = \{i : B_i \cap S \neq \emptyset\}$, then clearly $S \subseteq \cup_{i\in I} B_i$. On the other hand, $N(S) = \cup_{i\in I} A_i$. But then the condition of the problem gives us what we want:

$$|N(S)| = |\cup_{i\in I} A_i| \geq \sum_{i\in I} d_i = |\cup_{i\in I} B_i| \geq |S|.$$

Problem 4 (a): Given a matching M in G, let us call a path in G *alternating* if it alternates between edges from M and outside M. We want to show that if G satisfies Hall's condition and M does not cover A then there is an alternating path connecting two unmatched vertices $a \in A$ and $b \in B$.

We define the following orientation of G: edges in M are oriented from B towards A, whereas edges outside M point from A to B. Let S be the set of vertices that can be reached from a fixed vertex $a \in A$ in this oriented graph. Then S is exactly the set of vertices s such that an alternating path between a and s exists. Let $A' = A \cap S$ and $B' = B \cap S$. We need to show that B' contains an unmatched vertex.

Note that any vertex in A' except a was reached through an edge from the matching, in particular, there is no edge in M between A' and $B - B'$. But clearly no non-matching edge

uv can exist between $A' \ni u$ and $B - B' \ni v$, because then v could be reached from a via u, so v would be in $S \cap B = B'$. This means that there is no edge between A' and $B - B'$, so $B' \subseteq N(A')$, and since S is connected, in fact we have $N(A') = B'$.

Now by Hall's condition, we know that $|B'| = |N(A')| \geq |A'|$, and as A' contains all the "pairs" of the matched vertices in B' *plus* a, there has to be an unmatched vertex in B', as well. This shows the existence of an augmenting path.

Now starting with the empty matching, we can iteratively find augmenting paths and extend the current matching along it, until we reach a matching covering A. This proves Hall's theorem.