Graph Theory

Solutions 6

The aim of the homework problems is to help you understand the theory better by actively using it to solve exercises. ***Do not read the solutions*** *before you believe you have solved the problems: it ruins your best way of preparing for the exam. The purpose of this write-up is merely to provide some guideline on how solutions should look like, and to help clean up hazy arguments. For hints, feel free to consult your teaching assistant.*

Problem 3: Let G be a planar graph. We already know that G is bipartite iff all its cycles have even length (first exercise sheet), so it is enough to prove that the statement "every face has even length" is equivalent to these. We will first show that bipartite implies that all faces have even length, and then that if all faces have even length then all cycles have even length as well.

To see the first statement, let us take any face and a vertex v on it. Now start walking from v along the boundary of the face. Since the graph is bipartite, we alternately encounter vertices from the two classes, and when we reach v again, we are back at its class. So we made an even number of steps, hence the length of the face is even.

For the second statement, assume that all faces have even length, and consider a cycle in the drawing, of length l. The region bounded by the cycle is the union of faces, let the length of those faces be $l_1, \ldots, l_k$. Then the sum $\sum_{i=1}^{k} l_i$ counts each edge the cycle once, and each edge inside the region bounded by the cycle twice (once for both adjacent faces). $\sum_{i=1}^{k} l_i \equiv l \pmod 2$, and since all the l_i are even, l is also even.

Problem 5: Take an arbitrary (not necessarily proper) 3-coloring of G, and let a_i be the number of 2-colored sides of the i'th face. Note that if the vertices of this i'th triangle get 3 different colors, then $a_i = 3$, if they only get 2 different colors, then $a_i = 2$, and if all the vertices have the same color, then $a_i = 0$.

The sum $\sum a_i$ counts each 2-colored edge twice (once for each adjacent face), and since the a_i's can only take the values 0, 2 or 3, this means we have an even number of 3's among them. That means an even number of 3-colored triangles.

Problem 6: Draw a straight line segment between any pair of points in S that have distance exactly 1. It is enough to show that there are no crossing segments. Indeed, if that is the case, then the drawing corresponds to a plane embedding of a graph, where two points are connected by an edge if they have distance 1. Since a plane graph on n vertices has at most $3n-6$ edges, we get that there are at most $3n-6$ such pairs of points.

Suppose for contradiction that two of the above segments AB and CD intersect at some point X. Remember, both segments have length $l(AB)=l(CD)=1$. By the triangle inequality, $l(AX)+l(XC)\geq l(AC)$ and $l(XB)+l(DX)\geq l(BD)$, so

$$l(AC)+l(BD)\leq l(AX)+l(XB)+l(XC)+l(DX)=l(AB)+l(CD)=2.$$

But by the condition, $l(AC)$ and $l(BD)$ both are at least 1, so we in fact have equality everywhere. Similarly we get that $l(AD)$ and $l(BC)$ are 1, so we have four points in the plane, any two of unit distance, which is impossible. So indeed there is no crossing.