Graph Theory

Solutions 7

The purpose of this write-up is merely to provide some guideline on how solutions should look like, and to help clean up hazy arguments. For hints, feel free to consult your teaching assistant.

Problem 3:

We need to color the vertices of G using $m = \max \chi(G_r) + \chi(G_{r+1})$ colors. The main idea that we need is the fact that there is no edge between G_r and G_{r-2} for any r, so we can color them independently of each other.

We color the vertices inductively. First we color $G_0 = v$ with some color. Then we have $m - \chi(G_0)$ colors left to use for G_1. Since $\chi(G_1) \leq m - \chi(G_0)$, we can properly color G_1 using $\chi(G_1)$ colors that all differ from the colors of G_0. And so on: in the r'th step we color G_r using $\chi(G_r)$ colors that are all different from the ones we used to color G_{r-1} (we can do this because $\chi(G_r) \leq m - \chi(G_{r-1})$. And this will be a proper coloring, because all edges of G either go within some G_r, or between some G_{r-1} and G_r. (And, since G is connected, we color all the vertices eventually.)

Problem 4:

If G is l-degenerate, then $\chi(G) \leq l+1$. So suppose $\chi(G) > l+1$. Then G is not l-degenerate, hence contains some subgraph of minimum degree at least $l+1$. But as we saw in the first week of the course, a graph of minimum degree $l+1$ contains a path of length $l+1$. Contradiction.

Problem 5: The chromatic number of G is the minimum number of independent sets covering the vertices, so it is the minimum number of complete graphs in $\overline{G}$ covering all the vertices. As $\overline{G}$ is bipartite, its cliques are singletons and edges — to have as few cliques as possible, we want to use as many edges as we can. This means that $\chi(G) = n - \nu(\overline{G})$, where n is the number of vertices.

$\omega(G) = \alpha(\overline{G})$, so we want to have a large independent set. But an independent set I spans no edges, so $V - I$ covers all the edges. This means that large independent sets correspond to small edge-covering sets, i.e. $\alpha(\overline{G}) = n - \tau(\overline{G})$. But from Kőnig's theorem, we know $\nu(\overline{G}) = \tau(\overline{G})$ for bipartite $\overline{G}$, so we are done.