Graph Theory

Solutions 8

The purpose of this write-up is merely to provide some guideline on how solutions should look like, and to help clean up hazy arguments. For hints for other problems, feel free to consult your teaching assistant.

Problem 4:

Suppose u is a cut vertex and let v, w be two vertices adjacent to v, but in different blocks of G. By Kőnig's theorem, the edge set of G splits into perfect matchings, let M_v be the one containing uv and M_w be the one containing uw. Now look at $G' = G - u$, and let G_v be the component of v in it. $M_v - uv$ is a matching in G' that covers everything but v. This means that G_v contains an odd number of vertices. On the other hand, $M_w - uw$ is a matching in G' that covers everything but w, in particular, all the vertices in G_v. Hence G_v has an even number of vertices. This contradiction shows that the graph contains no cut vertex, i.e. it is 2-connected.

Problem 5:

By Vizing's theorem, the edge set of G can be partitioned into $d = \Delta + 1$ matchings $M_1, \ldots, M_d$ (one of them might be the empty matching). We will show that if for some i and j the sizes of M_i and M_j are at least two apart, say $e(M_i) \geq e(M_j) + 2$, then we can rearrange the edges of $M_i \cup M_j$ into two matchings M_i' and M_j' such that $e(M_i') = e(M_i) - 1$ and $e(M_j') = e(M_j) + 1$. This is enough, because e.g. $\sum e(M_l)^2$ will decrease, so repeating this operation, we will eventually arrive at a matching decomposition, where the sizes are at most one apart (so their value is either $\lfloor e/d \rfloor$ or $\lceil e/d \rceil$.

To prove this, look at $M_i \cup M_j$: it is a union of alternating cycles and paths. Since M_i has more edges than M_j, there must be a path that starts and ends with an edge from M_i. Flipping the edges along this path decreases the number of edges in M_i by 1 and increases the number in M_j by 1, and of course the new M_i and M_j are still matchings. So we can choose them to be M_i' and M_j' above.