Graph Theory

Solutions 9

The purpose of this write-up is merely to provide some guideline on how solutions should look like, and to help clean up hazy arguments. For hints for other problems, feel free to consult your teaching assistant.

Problem 3:

We prove by induction on r. For $r = 1$ the statement trivially holds. Now suppose we know it for $r - 1$, and take an arbitrary assignment of lists to the vertices of G. G is r-partite with parts $A_1, \ldots, A_r$, each of size 2. Suppose the vertices in some part share a common color on their lists. Then we assign this color to both and delete it from the list of every other vertex. By induction we can color the remaining $2(r-1)$ vertices from the new lists, this gives a proper coloring of G.

So we can assume that each pair of vertices from the same part has disjoint lists of colors. In some sense this means that we can assign the colors in A_i "independently" of the other. So let us build a bipartite graph between the vertices of G and the set of colors from the lists, where we connect a vertex to a color if the color appears on the list of the vertex. Our aim is to show that there is a matching from the set of vertices into the set of colors — this gives a proper coloring from the prescribed lists (in fact, all the colors will be different).

We check Hall's condition to prove this. So take a set S of vertices. Its neighborhood $N(S)$ in the bipartite graph is the union of the color lists of S. If $1 \leq |S| \leq r$ then clearly $|N(S)| \geq r \geq |S|$: any single vertex has r colors on its list. On the other hand, if $r+1 \leq |S| \leq 2r$ then S contains both vertices from some class A_i (pigeonhole principle). By our assumption, the lists in A_i are disjoint, so $|N(S)| \geq |N(A_i)| \geq 2r \geq |S|$. This proves that Hall's condition holds for any non-empty set S, hence a matching exists.