Exercise sheet 1

Due date: 13:00, March 7.

Location: Next to HG G 52.1.

Exercise 1.1 Let $\succeq$ be a preference order on $\mathcal{C}$ satisfying axioms (P1)-(P4). A function $\mathcal{U} : \mathcal{C} \to \mathbb{R}$ is called a *utility functional representing* $\succeq$ if

$$c \succeq c' \quad \Longleftrightarrow \quad \mathcal{U}(c) \geq \mathcal{U}(c').$$

(a) Show that all $\mathcal{U}$ representing $\succeq$ must be *quasiconcave*, i.e., for all $c, c' \in \mathcal{C}$ and $\lambda \in [0, 1]$,

$$\mathcal{U}(\lambda c + (1 - \lambda)c') \geq \min\{\mathcal{U}(c), \mathcal{U}(c')\}.$$

(b) Which axioms are needed for this result?

(c) Show by a counterexample that a preference order can be represented by a utility functional which is not concave.

Exercise 1.2 Suppose $\mathcal{D}$ is complete. Show that $B(e, \pi) = \mathcal{C}$ for all e if and only if there exists arbitrage of the second kind.

Exercise 1.3 Let $\succeq$ be a preference relation on $\mathcal{C}$. The goal is to show that $\succeq$ has a continuous, numerical representation (utility functional).

(a) Denote by $\mathbf{1}$ the vector with all components equal to 1. Prove that for each $c \in \mathcal{C}$ there exists a unique $\alpha(c) \in \mathbb{R}$ such that $\alpha(c)\mathbf{1} \sim c$.

(b) Define $\mathcal{U}(c) = \alpha(c)$ and show that $\mathcal{U}$ is a continuous, numerical representation of $\succeq$.

An increasing preference relation on $\mathcal{C}$ is called *homothetic* if

$$c \sim c' \Longrightarrow \beta c \sim \beta c', \quad \forall \beta \geq 0.$$

(c) Show that $\succeq$ is homothetic if and only if it admits a utility functional $\mathcal{U}$ that is homogeneous of degree one, i.e., $\mathcal{U}(\beta c) = \beta \mathcal{U}(c)$ for all $\beta \geq 0$.