Exercise sheet 10

Due date: 13:00, May 17.

Location: Next to HG G 52.1.

Exercise 10.1 For probability measures $Q \ll P$ the *(relative) entropy* of Q with respect to P is defined as

$$H(Q|P) = E_P\left[\frac{\mathrm{d}Q}{\mathrm{d}P} \ln \frac{\mathrm{d}Q}{\mathrm{d}P}\right] = E_Q\left[\ln \frac{\mathrm{d}Q}{\mathrm{d}P}\right].$$

In this problem, we consider the trinomial market introduced in Exercise 2.3 with $m = r = 0$, $u = -d$, $\pi^1 = 1$ and $p^i = P[S_1^1 = 1 + i]$.

(a) Find the measure Q^* minimizing the relative entropy $H(Q|P)$ over all equivalent martingale measures Q.

(b) Find the strategy ϑ^* maximizing expected utility of final wealth, with initial wealth 0 and exponential utility with parameter α, i.e.,

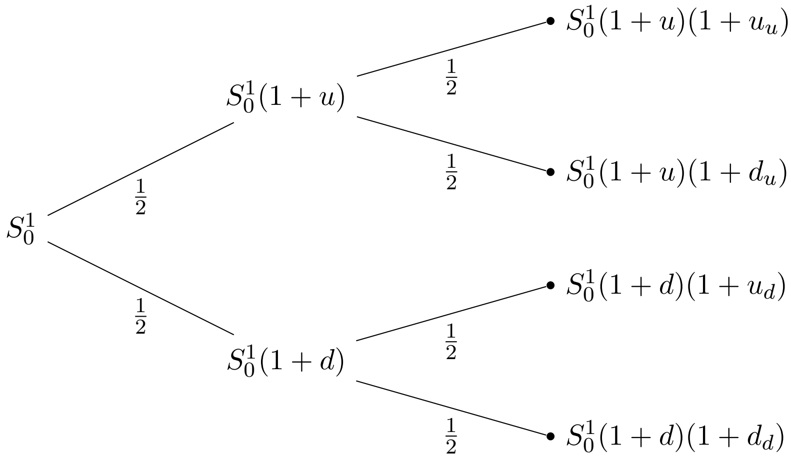
$$U_w(x) = 1 - e^{-\alpha x} \quad \text{and} \quad U_c(x) = 0.$$

Verify that

$$\frac{\mathrm{d}Q^*}{\mathrm{d}P} = \frac{e^{\eta^* \cdot \Delta X_1}}{E[e^{\eta^* \cdot \Delta X_1}]},$$

with $\eta^* = -\alpha \vartheta^*$.

Exercise 10.2 Consider the asset given by the following tree:



where the fractions denote probabilities. Let $S_k^0 = (1 + r)^k$ and

$$-1 < r \in (d, u) \cap (d_u, u_u) \cap (d_d, u_d) \neq \emptyset,$$

i.e., the market is arbitrage-free.

Strategies here can be identified with vectors in $\mathbb{R}^3$ via $\vartheta = (\vartheta_1, \vartheta_2^u, \vartheta_2^d)$. Find the optimizer ϑ^* to the problem of maximizing (exponential) utility of final wealth:

$$\max_{\vartheta \in \mathbb{R}^3} E\left[1 - \exp\left(-v_0 - G_2(\vartheta)\right)\right].$$

Exercise 10.3 Consider a general arbitrage-free single-period market.Fix x and let $U : [0,\infty) \to \mathbb{R}$ be a concave, increasing (utility) function, continuously differentiable on $(0,\infty)$, such that

$$\sup_{\vartheta \in \mathcal{A}(x)} E[U(x + \vartheta \cdot \Delta X_1)] < \infty,$$

with

$$\mathcal{A}(x) = \{\vartheta \in \mathbb{R}^d | x + \vartheta \cdot \Delta X_1 \geq 0 \, P\text{-a.s.}\}.$$

Furthermore, assume that the supremum is attained in an interior point ϑ^* of $\mathcal{A}(x)$.

(a) Show that

$$U'(x + \vartheta^* \cdot \Delta X_1)|\Delta X_1| \in L^1(P)$$

and the *first order condition*

$$E[U'(X + \vartheta^* \cdot \Delta X_1)\Delta X_1] = 0.$$

Hint: You may use that

$$y \mapsto \frac{U(y) - U(z)}{y - z}, \quad y \in (0,\infty) \setminus \{z\}$$

is nonincreasing. By optimality, ϑ^* is better than $\vartheta^* + \varepsilon\eta$ for any $\eta \neq 0$ and $0 < \varepsilon \ll 1$; so take the difference of corresponding utilities, divide by ε and look at $\varepsilon \searrow 0$. Exploit the hint to see that this quantity is monotonic in ε.

(b) Show that Q given by

$$\frac{\mathrm{d}\bar{Q}}{\mathrm{d}P} = \frac{U'(x + \vartheta^* \cdot \Delta X_1)}{E[U'(x + \vartheta^* \cdot \Delta X_1)]}$$

is an equivalent martingale measure.