Exercise sheet 11

Due date: 13:00, May 23.

Location: Next to HG G 52.1.

Exercise 11.1 Consider the arbitrage-free market in T periods with a riskless bond with zero interest rate. Assume that $H \in L^0_+(\mathcal{F}_T)$ is an attainable claim and fix one EMM Q. Let U be an exponential utility function, $\mathcal{A} = \Theta$ be the set of predictable processes, and consider the two functions

$$u(x) = \max_{\vartheta \in \mathcal{A}} E[U(x + G_T(\vartheta))]$$

and

$$u_H(x) = \max_{\vartheta \in \mathcal{A}} E[U(x + G_T(\vartheta) - H)].$$

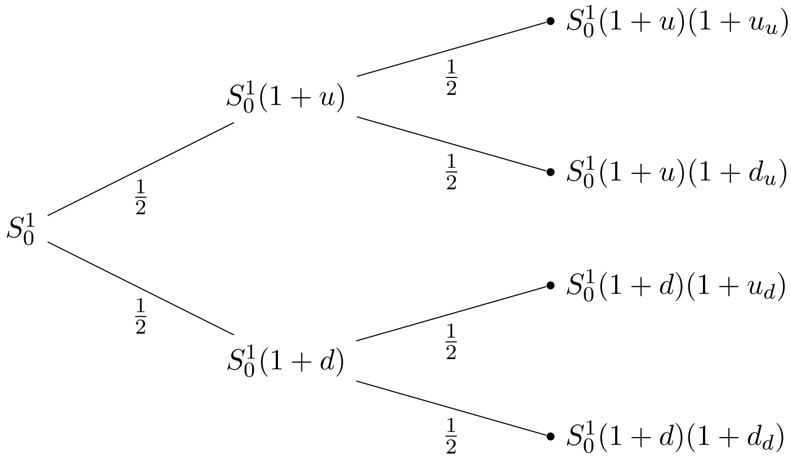
Given a wealth level x, the *utility indifference price* $p_H(x)$ of H is defined as the solution to

$$u(x) = u_H(x + p_H(x)).$$

(a) Show that $E_Q[H]$ is the unique solution to the above equation.

(b) Can the assumptions on the utility function be generalized?

Exercise 11.2 Recall the market structure from Exercise 10.2:



where the fractions denote probabilities. Let $S^0_k = (1+r)^k$ and

$$-1 < r \in (d,u) \cap (d_u, u_u) \cap (d_d, u_d) \neq \emptyset.$$

Strategies are identified with vectors in $\mathbb{R}^3$ via $\vartheta = (\vartheta_1, \vartheta_2^u, \vartheta_2^d)$. Find the optimizer ϑ^* to the problem

$$\max_{\vartheta \in \mathbb{R}^3} E\left[1 - \exp\left(-v_0 - G_2(\vartheta)\right)\right].$$

using dynamic programming.

Exercise 11.3 Consider an individual with endowment (income) $(e_0, \dots, e_T)$, $e_0 > 0$ and who only invests in the riskless bank account $S_k^0 = (1+r)^k$. Denote by W_k the wealth held in the bank account when leaving time point $k-1$ and $W_0 = v_0$. Then, for any consumption strategy,

$$W_{k+1} = (W_k + e_k - c_k)(1+r).$$

With $\beta \in (0,1]$, the individual aims to maximize

$$E\left[\sum_{k=0}^{T} \beta^k U(c_k)\right],$$

subject to

$$v_0 + \sum_{k=0}^{T} e^{-rk}(e_k - c_k) = 0,$$

over all (possibly negative) consumption plans, and where $\tilde{\beta} = 1/\beta - 1$ is a so-called *impatience parameter*. For simplicity, we will consider $\tilde{\beta} = r$. Assume that U is described by a quadratic parabola, so that U' is an affine function.[1]

Find the optimal consumption strategy.

[1] We make this assumption for the sake of the exercise, even though a parabola is not increasing on a sufficiently large domain.