Exercise sheet 12

Due date: 13:00, May 30.

Location: Next to HG G 52.1.

Exercise 12.1 Let $U : \mathbb{R} \to \mathbb{R}$ be a strictly increasing utility function and consider a general arbitrage-free market in T periods, with $\mathcal{F}_0$ trivial. Recall that $\mathcal{C} = G_T(\Theta) - L^0_+$.

(a) Show that an optimizer to

$$u(x) = \sup_{\vartheta \in \Theta} E\left[U(x + G_T(\vartheta)\right]$$

can be obtained from an optimizer of

$$u_{\mathcal{C}}(x) = \sup_{f \in \mathcal{C}} E\left[U(x + f)\right],$$

and vice versa.

(b) Denote by $\mathbb{P}_a$ the set of absolutely continuous martingale measures. Show that if Ω is finite and $f \in L^0$, then

$$f \in \mathcal{C} \iff E_Q[f] \leq 0, \quad \forall Q \in \mathbb{P}_a.$$

Exercise 12.2 Consider the market from Exercise 10.2 and 11.2. Let U be a utility function of power type, i.e.,

$$U(x) = \frac{x^\delta}{\delta},$$

for some $\delta < 0$. Denote by $u(x)$ the indirect utility of maximizing expected utility of final wealth

$$E[U(x + G_T(\vartheta))],$$

over x-admissible strategies ϑ. Formulate the dual problem and find an explicit expression for $u(x)$.

Hint: Calculate the *Legendre transform* J of U; there was a typo in the expression given on the blackboard.

Exercise 12.3 Consider a general market in T periods. Let $U : (0, \infty) \to \mathbb{R}$ be a strictly increasing and strictly concave utility function, and denote by u the indirect utility from maximizing the utility of final wealth:

$$u(x) = \sup_{\theta \in \Theta^x} E\left[U(x + G_T(\vartheta))\right],$$

for $x > 0$, where $\Theta^x = \{\vartheta \in \Theta | \vartheta \text{ is } x\text{-admissible}\}$.

(a) Assume that $u(x_0) < \infty$ for some $x_0 > 0$. Show that u is increasing, concave and $u(x) < \infty$ for all $x > 0$.

(b) Show that if U is unbounded from above and the market admits an arbitrage opportunity, then $u \equiv +\infty$.