Exercise sheet 2

Due date: 13:00, March 14.

Location: Next to HG G 52.1.

Exercise 2.1 Consider a market with a numéraire and the payoff matrix $\mathcal{D}$. Denote by π an equilibrium price vector at time 0. Let $\mathcal{D}'$ be a payoff matrix with the property

$$\operatorname{Im}\mathcal{D}' = \operatorname{Im}\mathcal{D}.$$

(a) Denote by $B(e^i, \pi; \mathcal{D})$ the budget set

$$\left\{ c \in \mathcal{C} \,\middle|\, \exists \vartheta \in \mathbb{R}^N \text{ with } c \leq e^i + \overline{\mathcal{D}}\vartheta \right\}.$$

Construct π' such that $B(e^i, \pi; \mathcal{D}) = B(e^i, \pi'; \mathcal{D}')$.

(b) Show that if $((e^1, \dots, e^I), \mathcal{D})$ has an equilibrium with consumption allocation $(c^1, \dots, c^I)$, then there exists an equilibrium for $((e^1, \dots, e^I), \mathcal{D}')$ with the same consumption.
Hint: Use the price vector π'.

Exercise 2.2 Consider the one-step *binomial market* described by

$$\pi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1+r & 1+u \\ 1+r & 1+d \end{pmatrix},$$

for some $r > -1$, u and d with $u > d$.

(a) Show that this market is free for arbitrage if and only if $u > r > d$.

(b) Construct an arbitrage opportunity for a market where $u = r > d$.

Exercise 2.3 Consider the one-step *trinomial market* described by

$$\pi = \begin{pmatrix} \pi^0 \\ \pi^1 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} \pi^0(1+r) & \pi^1(1+u) \\ \pi^0(1+r) & \pi^1(1+m) \\ \pi^0(1+r) & \pi^1(1+d) \end{pmatrix},$$

for some $r > -1$, u, m and d with $u > m > d$ and $u > r > d$.

(a) Show that $\mathbb{P}(D^0)$ is convex.
Note: The particular structure given in this exercise is not necessary.

(b) Calculate the set $\mathbb{P}(D^0)$ of equivalent martingale measures.
Hint: Use the probability of the 'middle outcome' as a parameter in a parametrization of $\mathbb{P}(D^0)$ as a line segment in $\mathbb{R}^3$.

(c) Denote by $\mathbb{P}_a(D^0)$ the set of all martingale measures Q which are absolutely continuous with respect to P, i.e., $Q \ll P$. An element $R \in \mathbb{P}_a(D^0)$ is an extreme point if $R = \lambda Q + (1-\lambda)Q'$ with $0 < \lambda < 1$ and $Q, Q' \in \mathbb{P}_a(D^0)$ implies $Q = Q'$, i.e., R cannot be written as a strict convex combination of elements in $\mathbb{P}_a(D^0)$.

Find the extreme points of $\mathbb{P}_a(D^0)$ and represent $\mathbb{P}(D^0)$ by writing it as a (strict) convex combination of such extreme points. Verify that this coincides with the answer found above.