Exercise sheet 3

Due date: 13:00, March 21.

Location: Next to HG G 52.1.

Exercise 3.1 Recall that $\mathbb{P}$ (resp. $\mathbb{P}_a$) denotes the set of all equivalent (resp. all absolutely continuous) martingale measures with the numéraire D^0. Consider an arbitrage-free market with numéraire D^0.

(a) Show that $\mathbb{P}_a = \overline{\mathbb{P}}$.

(b) Use (a) to show that for any random variable X,

$$\sup_{Q\in\mathbb{P}} E_Q[X] = \sup_{Q\in\mathbb{P}_a} E_Q[X].$$

(c) Show that for any payoff H, the supremum

$$\sup_{Q\in\mathbb{P}_a} E\left[\frac{H}{D^0}\right]$$

is attained in some $Q \in \mathbb{P}_a$. Does this imply that H is attainable?

Exercise 3.2 Let

$$\pi = \begin{pmatrix} 1 \\ 1100 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1.1 & 1320 \\ 1.1 & 1210 \\ 1.1 & 880 \end{pmatrix}.$$

This is the example with the gold market from the lecture, with three possible outcomes (compare to Exercise 2.3). Denote by H the payoff of a put option with strike $K = 900$, i.e.,

$$H = (900 - D^1)^+ = \begin{pmatrix} 0 \\ 0 \\ 20 \end{pmatrix}.$$

(a) Find

$$\sup_{Q\in\mathbb{P}(D^0)} E_Q\left[\frac{H}{D^0}\right].$$

(b) Find

$$\inf\{\pi\cdot\vartheta | \mathcal{D}\vartheta \geq H\}.$$

(c) Construct a market with $\mathbb{P}_a \neq \overline{\mathbb{P}}$, where we use the notation from Exercise 3.1.

Exercise 3.3 Let $(\mathcal{D}, \pi)$ be an arbitrage-free market with numéraire, H a payoff which is not attainable in $\mathcal{D}$ and $\pi_s(H)$ the seller's price for H, i.e.,

$$\pi_s(H) = \inf\{\vartheta \cdot \pi | \vartheta \in \mathbb{R}^N \text{ with } \mathcal{D}\vartheta \geq H\}.$$

Denote by $(\mathcal{D}^e, \pi^e)$ the extended market $(\mathcal{D}, H, \pi, \pi_s(H))$.

(a) Show that $(\mathcal{D}^e, \pi^e)$ always admits an arbitrage opportunity.

(b) Show that $(\mathcal{D}^e, \pi^e)$ does not admit an arbitrage opportunity of the second kind.