Exercise sheet 4

Due date: 13:00, April 4.

Location: Next to HG G 52.1.

Exercise 4.1

(a) Construct a market and a payoff H with $\pi_s(H) < \pi_b(H)$.

(b) Show that if $\pi_s(H) < \pi_b(H)$ for some payoff H, then neither $\pi_s(H)$ nor $\pi_b(H)$ is finite.

(c) Show that the existence of arbitrage is not sufficient for the property $\pi_s(H) < \pi_b(H)$ to hold for some payoff H.

Exercise 4.2 Consider a market with trading dates $k = 0, \ldots, T$, with N traded assets on the probability space $(\Omega, \mathcal{F}, P)$ and the filtration given by $\mathbb{F} = (\mathcal{F}_k)_{k=0,\ldots,T}$, i.e., a general multiperiod market.

For any strategy ψ, we define the process $\widetilde{C} = (\widetilde{C}_k)_{k=0,\ldots,T}$ by

$$\widetilde{C}_k(\psi) := \widetilde{V}_k(\psi) - \widetilde{G}_k(\psi).$$

(a) Show that

$$\Delta\widetilde{C}_{k+1}(\psi) = \Delta\psi_{k+1} \cdot S_k,$$

for $k = 1, \ldots, T-1$.

(b) Show that ψ is self-financing if and only if

$$\widetilde{C}_k(\psi) = \widetilde{C}_0(\psi),$$

for $k = 1, \ldots, T$.

Hint: Be careful with definitions at the first time point.

Remark: The process $\widetilde{C}$ is called the *cost process* for ψ.

Exercise 4.3 We can generalize the model from Exercise 2.2 to multiple trading dates in the following manner. First fix $r > -1$ and let $S_k^0 = (1+r)^k$. Now define $S_0^1 = 1$ and

$$S_k^1 = \prod_{i=1}^{k} R_i^1, \quad k = 1, \ldots, T,$$

where the R_k^1 are i.i.d. and

$$P[R_k^1 = 1+u] = 1 - P[R_k^1 = 1+d] \in (0,1),$$

for $u > d$.

Suppose now, for the sake of the exercise, that $r = 0$, $u = 0.5$ as well as $d = -0.5$ and consider the strategy (V_0, ϑ) given by

$$\vartheta_k = \frac{1}{S^1_{k-1}} 2^k 1_{\{k \le \tau\}},$$

where $\tau = \inf\{k | R^1_k = 1 + u\} \wedge T$.

(a) Calculate the biggest loss over all time points to see how it depends on T. Conclude that the strategy would not be admissible if $T = \infty$.

(b) Suppose that $T = \infty$ and calculate the value of the strategy at the stopping time τ.

Remark: The mathematical term 'martingale' has one of its origins in this type of strategy, also called martingales.