Exercise sheet 9

Due date: 13:00, May 9.

Location: Next to HG G 52.1.

Exercise 9.1 Recall that an investment and consumption pair $(\psi, \tilde{c})$ is self-financing if $\psi_1 \cdot S_0 + \tilde{c} = \tilde{v}_0$ and

$$\Delta\psi_{k+1} \cdot S_k + \tilde{c}_k = 0,$$

for $k = 1, \dots, T-1$. Define the undiscounted wealth by $\tilde{W}_0 = \tilde{v}_0$ and $\tilde{W}_k := \psi_k \cdot S_k$, $\tilde{W}_0 := \tilde{v}_0 = \psi_1 \cdot S_0 + \tilde{c}_0$, for $k = 1, \dots, T$, $W = \tilde{W}/S^0$ and $c = \tilde{c}/S^0$.

(a) Show in detail that $(\psi, \tilde{c})$ is self-financing if and only if

$$W_k = v_0 + \sum_{j=1}^{k} (\vartheta_j \cdot \Delta X_j - c_{j-1}), \quad \text{for } k = 0, \dots, T.$$

(b) Show that the pair $(\psi, \tilde{c})$ with initial wealth $\tilde{v}_0$ is self-financing if and only if

$$\tilde{W}_k = \tilde{v}_0 + \sum_{j=1}^{k} \vartheta_j \cdot \Delta S_j - \sum_{j=0}^{k-1} \tilde{c}_j.$$

Exercise 9.2 For a twice differentiable utility function $U : [0, \infty) \to \mathbb{R}$, the so-called *relative risk aversion* is given by

$$-\frac{xU''(x)}{U'(x)}.$$

(a) Characterize all utility functions $U = U^\gamma$ with constant relative risk aversion equal to γ. Normalize the functions so that $U^\gamma(1) = 0$ and $(U^\gamma)'(1) = 1$.

(b) Verify that $\lim_{\gamma \to 1} U^\gamma(x) = U^1(x)$ for all x.

(c) For a differentiable function $f : [0, \infty) \to [0, \infty)$, the *elasticity* of f is defined as

$$\frac{xf'(x)}{f(x)}.$$

Show that with $U^\gamma(0) = 0$ instead of the normalization above, utility functions with constant relative risk aversion $\gamma \neq 1$ also have constant elasticity.

Exercise 9.3 Consider a general arbitrage-free market with a numéraire, positive asset prices and $T = 1$. Denote by H^{call} the (discounted) payoff of a call option with discounted strike $K > 0$, i.e.,

$$H^{\text{call}} = \left(X_1^1 - K\right)^+.$$

(a) Show that

$$\left(X_0^1 - K\right)^+ \leq \pi_b(H^{\text{call}}) \leq \pi_s(H^{\text{call}}) \leq X_0^1.$$

Now consider the market introduced in Exercise 7.3, i.e., $(\Omega, \mathcal{F}, P)$ with $\mathcal{F}_0$ trivial, $\mathcal{F} = \mathcal{F}_1 = \sigma(S_1^1)$ and assets given by

$$\begin{aligned} S_0^0 &= 1, & S_0^1 &= 1, \\ S_1^0 &= e^r, & S_1^1 &= e^Y, \end{aligned}$$

where Y follows a standard normal distribution under P. Define the set

$$\Pi^{\text{bin}}(H^{\text{call}}) = \left\{ E_{P^b}\left[H^{\text{call}}\right] \,\middle|\, \begin{array}{l} P^b \circ (S_1^1)^{-1} \text{ has mass} \\ \text{in two points and} \end{array} \; E_{P^b}\left[\frac{S_1^1}{e^r}\right] = S_0^1 \right\}$$

as the set of arbitrage-free prices under measures for which the market is binomial.

(b) Construct a sequence of martingale measures absolutely continuous to P that converges weakly to a martingale measure P^b such that the law of S_1^1 under P^b has mass in only two points.

(c) Show that
$$\Pi^{\text{bin}}(H^{\text{call}}) \subseteq \left[\pi_b(H^{\text{call}}), \pi_s(H^{\text{call}})\right].$$

(d) Show that

$$\sup \Pi^{\text{bin}}(H^{\text{call}}) = 1 \quad \text{and} \quad \inf \Pi^{\text{bin}}(H^{\text{call}}) = (1-K)^+,$$

and conclude that the universal bounds in (a) are attained in this market.