Sheet 1

1. Let G be a topological group and let (X, μ) be a σ-compact, locally compact, metric space with a positive Radon measure μ and assume that G acts on X continuously such that $g_*\mu \ll \mu \ll g_*\mu$ for all $g \in G$. For given g denote by $\frac{\mathrm{d}g_*\mu}{\mathrm{d}\mu} : X \to [0, \infty]$ the Radon-Nikodym derivative of $g_*\mu$ with respect to μ.

a) Prove that the action $G \curvearrowright L^2_\mu(X)$ given by:

$$\pi_g f(x) := \sqrt{\frac{\mathrm{d}g_*\mu}{\mathrm{d}\mu}(x)} f(g^{-1}x)$$

is indeed an action, i.e. $\pi_{gh} = \pi_g \circ \pi_h$. What does this mean for the Radon-Nikodym derivative?

b) Assume that there exist a full-measure subset X' such that the family $\{\rho_x\}_{x \in X'}$ of functions from G to $[0, \infty]$ defined by $\rho_x(g) := \frac{\mathrm{d}g_*\mu}{\mathrm{d}_\mu}(x)$ is equicontinuous at the identity. Prove that π defines a unitary representation of G on $L^2_\mu(X)$.

2. Prove that every irreducible unitary representation of an abelian group is one-dimensional.

3. Find a natural homomorphism of a group G into the group of unitary operators on $L^2(\mathbb{R})$ which is strongly continuous but not uniformly continuous.

4. Let G be a locally compact, σ-compact, metric group and let m_G be a left-Haar measure on G. Recall that $L^1_{m_G}(G)$ is a Banach algebra, when equipped with the product:

$$\psi * \phi(g) := \int_G \psi(x)\phi(x^{-1}g)\,\mathrm{d}m_G(x)$$

Let π be a unitary representation of G. Prove that $\mathcal{H}_\pi$ is a module for the Banach algebra $L^1_{m_G}(G)$.