Sheet 2

1. Let G a locally compact abelian group, for all $f \in L^1(G)$ let $\check{f} : \hat{G} \to \mathbb{C}$ be the Fourier back-transform given by:

$$\check{f}(\xi) = \int_G \langle g, \xi \rangle f(g) \, \mathrm{d}g$$

As in the real case, the restriction to $L^1(G) \cap L^2(G)$ defines an isometry with range in $L^2(\hat{G})$ with respect to a suitably normalized Haar measure. This extends to a unitary isomorphism $L^2(G) \cong L^2(\hat{G})$ which is G-equivariant.
For the representation of G on $L^2(\hat{G})$ determine the spectral measures, the projection valued measures and the measureable functional calculus.

2. Let π be a unitary representation of the locally compact abelian group G, $v \in \mathcal{H}_\pi$ and assume that $\chi_0 \in \hat{G}$. Denote by $\mathcal{H}_\pi(v)$ the cyclic subrepresentation generated by v. Prove that the following are equivalent:

1. $\mu_v(\{\chi_0\}) > 0$
2. There is $w \in \mathcal{H}_\pi(v) \setminus \{0\}$ such that $\pi_g w = \chi_0(g) w$.

3. Prove that the Heisenberg group and the "$ax + b$"-group are amenable.

4. Prove that for a group G (with the standing assumptions) the following are equivalent:

1. G is amenable and has property (T)
2. G is compact

5. Prove that $\mathrm{SL}_2(\mathbb{R})$ has no non-trivial, finite dimensional, unitary representations, once using the Mautner phenomenon and once using the Howe-Moore theorem.

6. Prove the Mautner phenomenon for $\mathrm{SL}_3(\mathbb{R})$.