Sheet 3

1. Prove the Mautner phenomenon for $\mathrm{SL}_3(\mathbb{R})$.

2. Prove that for a group G (with the standing assumptions) the following are equivalent:

1. G is amenable and has property (T)
2. G is compact

3. Let π be a unitary representation of a group G. Let $\mathcal{H}_\pi^*$ be the dual space of $\mathcal{H}_\pi$ and define a representation $\overline{\pi}$ of G on $\mathcal{H}_\pi^*$ by inverse transpose, i.e.:

$$\overline{\pi}_g \lambda(v) = \lambda(\pi_{g^{-1}} v) \quad \forall v \in \mathcal{H} \forall g \in G \forall \lambda \in \mathcal{H}_\pi^*$$

The representation $\overline{\pi}$ is called the *contragredient* of π.

a) Prove that $\overline{\pi}$ is a unitary representation of G.

b) Prove that the regular representation of G is isomorphic to its contragredient.

c) Formulate a sufficient criterion π has to satisfy so that $\pi \cong \overline{\pi}$.

4. In what follows, π will always denote an irreducible unitary representation of G, and $[\pi]$ is the class of unitary representations of G isomorphic to π.

a) Find (a group G and) a cyclic representation ρ of G such that π has multiplicity more that 1 in ρ. *Hint:* The dihedral group is assumed to give a simple example.

b) Find an upper bound for the multiplicity of π in a cyclic representation ρ.