# Sheet 4

**1.** For the Lie group $G = \mathrm{SU}(2)$, determine the space of conjugacy classes:

$$\mathrm{SU}(2)^\sharp := {}^{\mathrm{SU}(2)}\!/\!_{\sim}$$

where $g \sim h$ iff there is $x \in G$ such that $xgx^{-1} = h$. Equip $L^2(\mathrm{SU}(2)^\sharp)$ with the push forward of the Haar measure under the quotient map and determine an orthonormal basis for that space.

**2.** Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$, and $\pi$ a representation of $G$. For a given vector $v \in \mathcal{H}_\pi$ and $x \in \mathfrak{g}$, we say that $v$ has a weak derivative for $x$, if there is $v_x \in \mathcal{H}_\pi$ such that for all $\mathscr{C}^1$-smooth vectors $w \in \mathcal{H}_\pi$ holds:

$$\langle v_x, w\rangle = -\langle v, \pi(x)w\rangle$$

a) Assuming $v$ has weak partial derivatives for a basis of $\mathfrak{g}$, prove that $v$ is $\mathscr{C}^1$.

b) Prove that the total derivative $T$ attached to $\pi$ is a closed operator.

**3.** Equip the Lie algebra $\mathfrak{su}(2)$ of the real Lie group $\mathrm{SU}(2)$ with the inner product:

$$\langle X, Y\rangle = \frac{1}{2}\operatorname{tr} XY$$

Let $\pi$ be an irreducible representation of $\mathrm{SU}(2)$. Calculate the Casimir operator $\Omega = -\overline{T}^*\overline{T}$ for $\pi$.