Sheet 5

1. Prove that $\mathrm{SL}_3(\mathbb{R})$ has property (T) using effective decay of matrix coefficients.
Hint: Let $\varphi \in C_c^\infty\big(\mathrm{SL}_3(\mathbb{R})\big)$ non-negative, with integral 1 and look at the self-adjoint contraction $\pi(\varphi)^*(\pi_{a_t} + \pi_{a_t}^{-1})\pi(\varphi)$.

2. Let $p \geq 3$ prime. Show that every non-trivial, finite dimensional, complex representation of $\mathrm{SL}_2(\mathbb{F}_p)$ has dimension at least $\frac{p-1}{2}$.
Hint: $\frac{p-1}{2}$ is the number of squares in $\mathbb{F}_p$.

3. Let G be a finite group and $H = \mathbb{R}^2 \rtimes G$. Assume that for some $t \in \hat{\mathbb{R}}^2$ non-trivial holds $G_t = \{g \in G; \hat{\theta}_g(t) = t\}$ is non-trivial. Let ρ be any irreducible, unitary representation of G_t. Construct a representation π of H such that for the eigenspace V_t of $\mathbb{R}^2$ for the character t holds $\pi_{G_t}\big|_{V_t} \cong \rho$.

4. Let $A := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, let $G := \mathbb{Z}^2 \rtimes \mathbb{Z}$, where $(1,0)$ acts on $\mathbb{Z}^2$ by A. Note that $\widehat{\mathbb{Z}^2} \cong \mathbb{T}^2$. Show that the following construction yields an (irreducible) unitary representation of G:
Fix an A-invariant, σ-finite (ergodic) measure on T^2, and let $\mathcal{H}_\mu := L^2(\mathbb{T}^2, \mu)$, with the action of G on $\mathcal{H}_\mu$ by the multiplication action for $\mathbb{Z}^2$ and the measure-preserving Koopman operator for $\mathbb{Z}$.

The exercises in this sheet will not feature among those examined in the final exam.