Sheet 6

1. Let G be a finite group and $H = \mathbb{R}^2 \rtimes G$. Assume that for some $t \in \hat{\mathbb{R}}^2$ non-trivial holds $G_t = \{g \in G; \hat{\theta}_g(t) = t\}$ is non-trivial. Let ρ be any irreducible, unitary representation of G_t. Construct a representation π of H such that for the eigenspace V_t of $\mathbb{R}^2$ for the character t holds $\pi_{G_t}\big|_{V_t} \cong \rho$.

2. Let $A := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, let $G := \mathbb{Z}^2 \rtimes \mathbb{Z}$, where $(1,0)$ acts on $\mathbb{Z}^2$ by A. Note that $\widehat{\mathbb{Z}^2} \cong \mathbb{T}^2$. Show that the following construction yields an (irreducible) unitary representation of G:
Fix an A-invariant, σ-finite (ergodic) measure on T^2, and let $\mathcal{H}_\mu := L^2(\mathbb{T}^2, \mu)$, with the action of G on $\mathcal{H}_\mu$ by the multiplication action for $\mathbb{Z}^2$ and the measure-preserving Koopman operator for $\mathbb{Z}$.

3. Recall that for a given character χ on A extended to $B = MAN$, the principal series representation corresponding to the character χ is its induced representation on $G = \mathrm{SL}_2(\mathbb{R})$.

a) Prove that the resulting representation is indeed unitary.

b) Prove that the representation is irreducible unless the character used in the induction procedure is trivial on A and non-trivial on M. Prove that in the latter case the resulting representation is the sum of two mock discrete series.

c) Prove that matrix coefficients for these representations satisfy the same bound by the Harish-Chandra function as matrix coefficients for the regular representation of G and hence the principal series representations are tempered.

4. Recall that π_0 was the unitary representation of $\mathrm{SL}_2(\mathbb{R})$ obtained by induction of the trivial representation on B. Generalize this construction to the group $\mathrm{SL}_2(\mathbb{C})$ and calculate the Harish-Chandra spherical function for $\mathrm{SL}_2(\mathbb{C})$.

These exercises will not feature among those examined in the final exam.